Homework 11 Bonus #1

Let $\varphi(y)$ and $K(x,y)$ be continuous for $x, y \in [a,b]$ and assume $|K(x,y)| \le M$.

For any $x \in [a,b]$ and any $\lambda \in \mathbb{R}$ we claim the the equation

$$f(x) = \lambda \int_a^x K(x,y) f(y)\, dy + \varphi(x)$$

has a solution.

Key Idea: For the metric space $C([a,b])$ we use the *weighted* norm

$$\|u\|_* = \max_{[a,b]} |u(x) e^{-\gamma x}|,$$

where the constant $\gamma > 0$ will be chosen later. For $x \in [a,b]$ let

$$Af(x) = \lambda \int_a^x K(x,y) f(y)\, dy + \varphi(x).$$

We show that with this norm, A is a contracting map. For any f and g in $C([a,b])$

$$[Af(x) - Ag(x)| = \lambda \int_a^x K(x,y)[f(y) - g(y)]\, dy$$

so

$$[Af(x) - Ag(x)]e^{-\gamma x} = \lambda \int_a^x K(x,y)\Big([f(y) - g(y)]e^{-\gamma y}\Big) e^{\gamma(y-x)}\, dy.$$

Thus,

$$\begin{aligned}
|[Af(x) - Ag(x)]e^{-\gamma x}| &\le \lambda M \int_a^x \Big|[f(y) - g(y)]e^{-\gamma y}\Big| e^{\gamma(y-x)}\, dy \\
&\le \lambda M \|f - g\|_* \int_a^x e^{\gamma(y-x)}\, dy \\
&= \lambda M \|f - g\|_* \frac{e^{\gamma(y-x)}}{\gamma}\Big|_{y=a}^{y=x} \\
&= \frac{\lambda M \|f - g\|_*}{\gamma}[1 - e^{\gamma(a-x)}] \\
&\le \frac{\lambda M \|f - g\|_*}{\gamma}.
\end{aligned}$$

To make A contracting, pick $\gamma > \lambda M$.

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