Compactness and Continuity

Theorem *Let f be a continuous map of a compact metric space X into a metric space V. Then f is uniformly continuous on X.*

PROOF: This is only slightly different from the proof on Pag 91 of Rudin – but I hope it is clearer.

Since f is continuous at every point x of X, given any $\epsilon > 0$ there is an $r(x)$ so that

$$\text{if } \ d_X(x,y) < r(x)) \ \text{ then } \ d_V(f(x), f(y)) < \frac{1}{2}\epsilon. \tag{1}$$

The open balls centered at x and radius r are an open cover of X. The radii $r(x)$ are the raw ingredients for picking our δ.

At this point it is tempting to use the compactness of X to take a finite sub-cover of these balls. However, points in these ball may be almost $2r(x)$ (a diameter) apart. Thus, instead we use the open balls $B(x; \frac{1}{2}r(x))$ of half the radius. These are an open cover of X. Because X is compact, there is a finite sub-cover by open balls centered at, say, $p_1, p_2, \ldots, p_N$ and radii $\frac{1}{2}r(p_j)$. Thus,

$$\text{if } \ x \in X \quad \text{then for some } 1 \le j \le N \ \text{ we have } \quad d_X(x, p_j) < \tfrac{1}{2}r(p_j). \tag{2}$$

Now let δ be the smallest of these:

$$\delta = \min_j \tfrac{1}{2}r(p_j).$$

Say x and y are any points in X with $d_X(x,y) < \delta$. For this x there is at least one index j for which (2) holds. Therefore also

$$\begin{aligned} d_X(y, p_j) &< d_X(y,x) + d_X(x, p_j) \\ &\le \delta + \tfrac{1}{2}r(p_j) \le \ r(p_j) \end{aligned}$$

Consequently, using (1),

$$\begin{aligned} d_V(f(x), f(y)) \ &< d_V(f(x), f(p_j)) + d_V(f(y), f(p_j)) \\ &< \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon \ = \ \epsilon. \end{aligned}$$

Because this holds for any points in X with $d_X(x,y) < \delta$, this completes the proof that f is uniformly continuous in X.