Basic Examples

For each of these you should be able to determine if the set is:

finite   countable   bounded   open   closed   connected   compact

$A = \{x \in \mathbb{R} : x = 1, 2, 3, 4\}$

$B_1 = \{x \in \mathbb{R} : 0 < x < 1\}$

$B_2 = \{(x,\, y) \in \mathbb{R}^2 : 0 < x < 1,\ y = 0\}$

$C = \{x \in \mathbb{R} : 0 \le x \le 1\}$

$D = \{x \in \mathbb{R} : x = 1, 2, 3, 4, \ldots\}$

$E = \{x \in \mathbb{R} : x = 1, 1/2, 1/3, \ldots\}$

$F = \{x \in \mathbb{R} : x = 1, 1/2, 1/3, \ldots\} \cup \{0\}$

$G = \{x_n = (-1)^n + \dfrac{1}{n} \in \mathbb{R}, n = 1, 2, 3, \ldots\}$

$H = \{(x, y) \in \mathbb{R}^2 : x \text{ and } y \text{ are positive integers}\}$

$I = \{(x, y) \in \mathbb{R}^2 : x + y > 1\}$

$J = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < 4\}$

$K = \{e_1 = (1,\, 0,\, 0,\, 0),\ e_2 = (0,\, 1,\, 0,\, 0),$
$\quad e_3 = (0,\, 0,\, 1,\, 0,\,),\ e_4 = (0,\, 0,\, 0,\, 1)\ \in \mathbb{R}^4\}$

$\ell_2 = \{\text{sequences } x = (x_1,\, x_2,\, x_3,\, \ldots) \text{ where } \quad x_j \in \mathbb{R} \quad \text{and} \quad \sum_j |x_j|^2 < \infty\}$

Inner product: $\langle x,\, y\rangle = x_1y_1 + x_2y_2 + x_3y_3 + \cdots,$ Norm: $|x| = \sqrt{\sum_{i=1}^{\infty} |x_i|^2}\,.$

This is Hilbert's *Hilbert Space*

L In ℓ_2 let $e_1 = (1,\, 0,\, 0,\, ,0, \ldots),\ e_2 = (0,\, 1,\, 0,\, 0,\, \ldots),\ e_3 = (0,\, 0,\, 1,\, 0,\, \ldots),$ $e_4 = (0,\, 0,\, 0,\, 1,\, \ldots),\ \ldots$

If $i \ne j$ then $|e_i - e_j| = \sqrt{2}$. This set L is closed and bounded but not compact since if $0 < r < \sqrt{2}$ then the open balls $B_n = \{x \in \ell_2 : |x - e_n| < r\}$ cover this set but there is no finite sub-cover.

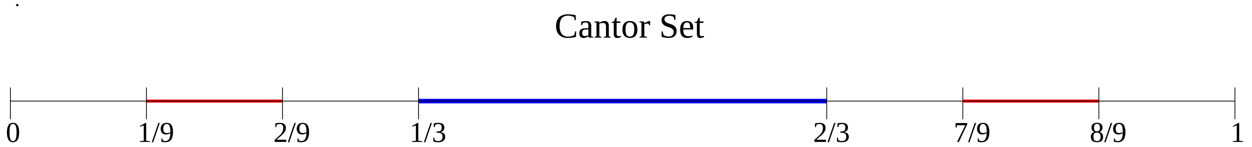
$\mathbb{Q} = \{x \in \mathbb{R} : x \text{ is a rational number}\}$

$\mathbb{Q}^2 = \{(x, y) \in \mathbb{R}^2 : x \text{ and } y \text{ are rational numbers}\}$

The Cantor Set This surprising example was found in 1874 and has greatly influenced mathematics, particularly after the work of Cantor.

Begin with the interval $J = [0, 1]$. Divide it into 3 equal segments and delete the middle piece, $U_1 = (\frac{1}{3}, \frac{2}{3})$. Divide the remaining two intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$ into three pieces and delete their middle pieces,

$$U_2 = (\tfrac{1}{9}, \tfrac{2}{9}) \cup (\tfrac{7}{9}, \tfrac{8}{9})$$



Divide the remaining four intervals into three pieces and delete their middle pieces:

$$U_3 = (\tfrac{1}{27}, \tfrac{2}{27}) \cup (\tfrac{7}{27}, \tfrac{8}{27}) \cup (\tfrac{19}{27}, \tfrac{20}{27}) \cup (\tfrac{25}{27}, \tfrac{26}{27})$$

Continue repeating this. The deleted set is

$$U := U_1 \cup U_2 \cup U_3 \cup U_4 \cup \cdots$$

The *Cantor Set* $K := J - U$ is what is left. Since each of the sets U_j are open, so is their union, U. There for K is a closed set. If you write the real numbers in $[0, 1]$ using base 3, then one only uses 0, 1, and 2 (much as base 10 we only use 0, 1, . . . ,9. the middle third intervals (Those in U) are precisely those whose base 3 representation have only 0's and 2's (no 1's). This set is is one-to-one correspondence with all the real numbers in $[0, 1]$ written base 2. Thus K *is uncountable.*

We can compute the length of the U_j.

The length of U_1 is $1/3$
The length of U_2 is $2/9$
The length of U_3 is $4/27$
The length of U_k is $2^{k-1}/3^k$.

Therefore the length of U is the sum of the geometric series $\sum_{k=1}^{\infty} \frac{2^{k-1}}{3^k} = 1$. Since the length of J is 1, we conclude that the length of the Cantor set K is $1 - 1 = 0$. Thus K is an uncountable set with measure 0!.

Hardly intuitive.

[DISCRETE METRIC On *any* set S you can define a quite crude metruc, the *discrete metric*. For p and q in S, if $p \neq q$ define $d(p, q) = 1$ while if $p = q$ define $d(p, p) = 0$. The axioms for a metric are easy to verify.

Because the ball of radius $1/2$ centered at p only has the one point $\{p\}$, each set consisting of one point is open. Since every set is the union of one point sets, *every set is open*. Every set in bounded, in fact, it is in a closed ball ball of radius 1 centered at any point p.

Because every point is isolated, there are no limit points. This implies that every set is closed (this also follows since every set is the complement of some set – and all sets are open – so their complements are closed.

The only compact sets are finite sets since most covers by open sets do not have a finite sub-cover. This thus gives examples of mertic spaces with closed and bounded sets that are not compact.