The Archimedean Property

Definition An ordered field F has the *Archimedean Property* if, given any positive x and y in F there is an integer $n > 0$ so that $nx > y$.

Theorem The set of real numbers (an ordered field with the Least Upper Bound property) has the Archimedean Property.

This is the proof I presented in class. It is one of the standard proofs. The key is the following Lemma.

Lemma The set $\mathbb{N}$ of positive integers $\mathbb{N} = \{0, 1, 2, \ldots\}$ is not bounded from above.

Proof Reasoning by contradiction, assume $\mathbb{N}$ is bounded from above. Since $\mathbb{N} \subset \mathbb{R}$ and $\mathbb{R}$ has the least upper bound property, then $\mathbb{N}$ has a least upper bound $\alpha \in \mathbb{R}$. Thus $n \leq \alpha$ for all $n \in \mathbb{N}$ and is the smallest such real number.
Consequently $\alpha - 1$ is *not* an upper bound for $\mathbb{N}$ (if it were, since $\alpha - 1 < \alpha$, then α would not be the *least* upper bound). Therefore there is some integer k with $\alpha - 1 < k$. But then $\alpha < k + 1$. This contradicts that α is an upper bound for $\mathbb{N}$.

Proof of the Theorem Since $x > 0$, the statement that there is an integer n so that $nx > y$ is equivalent to finding an n with $n > y/x$ for some n, But if there is no such n then $n < y/x$ for *all* integers n. That is, y/x would be an upper bound for the integers. This contradicts the Lemma.

Remark Homework Set 2 will have an example of an ordered field that does not have the Archimedean property.

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