Taylor's Theorem - Integral Remainder

Theorem Let $f : \mathbb{R} \to \mathbb{R}$ be a function that has $k+1$ continuous derivatives in some neighborhood U of $x = a$. Then for any $x \in U$

$$\begin{aligned} f(x) =& f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \\ &+ \frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x f^{(k+1)}(t)\frac{(x-t)^k}{k!}\,dt. \end{aligned}$$

Remark In this version, the error term involves an integral. Because of this, we assume that f^{k+1} is continuous, whereas previously we only assumed this derivative exists. However, we get the valuable bonus that this integral version of Taylor's theorem does not involve the essentially unknown constant c. This is vital in some applications.

Proof: For clarity, fix $x = b$. By the Fundamental Theorem of Calculus,

$$f(b) = f(a) + \int_a^b f'(t)\,dt.$$

We integrate by parts – with an intelligent choice of a constant of integration:

$$\begin{aligned} u =& f' & dv &= dt \\ du =& f''\,dt & v &= t-b \end{aligned}$$

Then

$$\begin{aligned} f(b) =& f(a) + f'(t)(t-b)\Big|_{t=a}^{t=b} - \int_a^b f''(t)(t-b)\,dt \\ =& f(a) + f'(a)(b-a) + \int_a^b f''(t)(b-t)\,dt. \end{aligned}$$

Repeat this integration by parts:

$$\begin{aligned} u =& f'' & dv &= (b-t)dt \\ du =& f'''\,dt & v &= -(b-t)^2/2 \end{aligned}$$

to find

$$\begin{aligned} f(b) =& f(a) + f'(a)(b-a) - f''(t)\frac{(b-t)^2}{2}\Big|_{t=a}^{t=b} + \int_a^b f'''(t)\frac{(b-t)^2}{2}\,dt \\ =& f(a) + f'(a)(b-a) + f''(a)\frac{(b-a)^2}{2} + \int_a^b f'''(t)\frac{(b-t)^2}{2}\,dt. \end{aligned}$$

To find the general formula we claimed, just repeat the integrations by parts. As an exercise, it is instructive to carry out one more step to obtain the formula for $k = 3$.