Dirichlet's Principle

By 1840 it was known that if $S \subset \mathbb{R}$ is a closed and bounded set and $f : S \to \mathbb{R}$ is a continuous function, then there are points p and q in S where f has its maximum and minimum value.

Mathematicians and physicists were considering more complicated functions, such as, on a smooth surface S in $\mathbb{R}^3$ finding the shortest path in the surface joining the two points p and q. If we write the curve as $\vec{\gamma}(t) = (x(t), y(t), z(t)) \subset S$ where $\vec{\gamma(0)} = p$ and $\vec{\gamma}(1) = q$, then the length of the curve is

$$J(\vec{\gamma}) = \int_0^1 |\vec{\gamma}\,'(t)|\, dt.$$

To find the curve minimizing the distance we need to look at all curves in the surface and find the curve minimizing J. Thus we seek functions $x(t)$, $y(t)$, and $z(t)$. If the surface is smooth, is there always a minimizing curve? If so, is it smooth?

Historically, the first interesting problem of this sort was to study a bead, starting from rest, sliding down a curve under the influence of gravity. In particular, given the points P and Q, find the equation of curve $y = f(x)$ from P to Q so the particle arrives at Q

in the least time. This is called the *Brachistochrone Problem*. The solution is interesting – and not at all obvious. Look it up.

One problem for a function $u(x, y)$ in several variables arose in a number of applications. Let $\Omega \subset \mathbb{R}^2$ be a bounded region with a smooth boundary $\partial\Omega$. and let $f(x, y)$ be a smooth function defined on the boundary, $\partial\Omega$. We seek a function $u(x, y)$ that minimizes the "energy"

$$J(v) = \iint_\Omega [v_x^2 + v_y^2]\, dx\, dy = \iint_\Omega |\nabla v|^2\, dx\, dy \quad (1)$$

among all functions $v(x, y)$ that agree with f on the boundary: $v(x, y) = f(x, y)$ for $(x,\, y) \in \partial\Omega$. In 1851, for his proof of what we call the Riemann Mapping Theorem, Riemann was seeking a minimizer since this minimizer it would be a solution of the Laplace equation:

$$u_{xx} + u_{yy} = 0 \quad \text{in} \;\; \Omega \quad \text{with} \;\; u = f \;\; \text{on } \partial\Omega. \quad (2)$$

It is easy to show this. Since we want to minimize something, the idea is to use that at a minimum of a real-valued function $\varphi(t)$ its first derivative is zero.

The computation we will use to seek a minimum of the (smooth) function $J(v)$ closely follows that used for a real-valued function $f(X)$ of several variables, $X = (x_1, x_2, \ldots, x_n)$. We recall this. Say f has a local minimum at an interior point X_0 of the set where f is defined. For any vector $Z \in \mathbb{R}^n$ let

$$\varphi(t) = f(X_0 + tZ).$$

Observe that $f(X_0 + tZ) \geq f(X_0) = \varphi(0)$, that is, $\varphi(0) \leq \varphi(t)$ for all t near zero. Thus φ has a local min at $t = 0$ so its derivative at 0 is zero: $\varphi'(0) = 0$ (this is the directional derivative of f at X_0 in the direction of the vector Z). But by the chain rule,

$$0 = \varphi'(0) = \nabla f(X_0) \cdot Z.$$

Because Z ia an arbitrary vector, this says that ∇f is orthogonal to all vectors so it must be zero, that is, $\nabla(f(X_0) = 0$.

We use the same procedure to find the minima of $J(v)$ in equation (1). Say a function u satisfying the boundary condition minimizes J. Let $h(x, y)$ be any smooth function that is zero on the boundary $\partial\Omega$. Then for any real t the function $u(x, y) + th(x, y)$ also satisfies the boundary conditions. Thus the function

$$\varphi(t) := J(u+th) = \iint_\Omega \left[|\nabla u|^2 + 2t\nabla u\cdot\nabla h + t^2|\nabla h|^2\right] dx\, dy$$

has a minimum at $t = 0$. Therefore $\varphi'(0) = 0$. That is,

$$\iint_\Omega \nabla u \cdot \nabla h \, dx\, dy = 0 \tag{3}$$

for *any* smooth function h that is zero on the boundary. Assuming the minimizer u is smooth,

$$\nabla u \cdot \nabla h = \nabla \cdot (h\nabla u) - h\Delta u,$$

where $\Delta u = \nabla \cdot \nabla u = u_{xx} + u_{yy}$ is the Laplacian. Thus integrating by parts (the Divergence Theorem), equation (3) implies that

$$\iint_\Omega (\Delta u) h \, dx\, dy = 0 \tag{4}$$

for all h that are zero on $\partial\Omega$. This implies that $\Delta u = 0$ throughout Ω (Proof: say $\Delta u > 0$ in a small disk $Q \subset \Omega$. Pick any h that is positive on this disk and zero outside it. But then for this h we have

$$\iint_\Omega (\Delta u) h \, dx\, dy = \iint_Q (\Delta u) h \, dx\, dy > 0.$$

contradicting equation (4). Thus, finding a minimizer of (1) gives a solution of the Laplace equation (2) with the desired boundary values.

Riemann's innovation was using the existence of a minimizer of (1) to prove the existence of a solution of the boundary value problem (2). He call this *Dirichlet's Principle.* Since in (1) $J(v)$ is bounded below (by zero), it is clear that J has an infimum among all functions v satisfying the boundary condition. It is not at all clear that there is a twice differentiable function u that actually minimizes J. To illustrate the difficulty Weierstrass gave an explicit example of a related problem

$$\text{Minimize } J(v) := \int_{-1}^{1} x^2 v'^{\,2}(x)\,dx$$

for all v with $v(-1) = -1$ and $v(1) = 1$. Following his reasoning, we show that J have an infimum but does not have a minimum. He considered the sequence of functions

$$v_n(x) = \begin{cases} -1 & \text{if } -1 \le x \le -1/n, \\ nx & \text{if } -1/n \le x \le 1/n.\,. \\ 1 & \text{if } 1/n \le x \le 1 \end{cases}$$

By an easy calculation has $J(v_n) = 2/(3n) \to 0$. This shows that the $\inf J(v) = 0$ (if you prefer a smooth sequence of functions you can use $v_n(x) := \frac{\tanh nx}{\tanh n}$). However, if there is a v with $J(v) = 0$ then $v' = 0$, so v must be a constant– and that can't satisfy the boundary conditions. Thus this J has an *inf* but not a *min*.

Mathematicians generally believed the idea behind Riemann's proof of the existence of a solution to (2) – but there certainly was a gap in the proof. It took about 50 years to develop the ideas such as compactness needed to understand the situation adequately.

TOY EXAMPLE: Here is a toy (but not obvious) example where the idea behind Dirichlet's Principle works immediately. Say you are seeking a solution (x, y) of the two equations

$$\begin{aligned} 2x(x^2 + y^2) + y - 1 = 0 \\ 2y(x^2 + y^2) + 2y^3 + x + 2 = 0 \end{aligned} \tag{5}$$

Idea: find a function $f(x, y)$ that has a local minimum somewhere and with the property that equations (5) are the equations $f_x = 0$ and $f_y = 0$, so

they will be satisfied at this local minimum.
Consider the function

$$f(x, y) = (x^2 + y^2)^2 + y^4 + 2xy - 2x - +4y - 3$$

Computing f_x and f_y, except for a factor of 2, these are exactly the equations (5) we wanted to solve. Thus, if we can show that f has a local minimum somewhere, then at least one solution exists, namely (x_0, y_0).
With this problem in mind, in Homework Set 2 Problem 4 you found a number R so that if $x^2 + y^2 \geq R^2$ then $f(x, y) \geq 1$. Since the disk $x^2 + y^2 \leq R^2$ is compact, there is at least one point (x_0, y_0) in this disk where f attains its minimum. Because $f(0, 0) = -3 < 1$, this point is not on the boundary of the disk so it is an interior point. Thus, at this point, the gradient of f is zero, that is, $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$.

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