Problem Set 8

DUE: Thurs. Nov. 11, 2010. *Late papers will be accepted until 1:00 PM Friday.*

Note: We say a function is *smooth* if its derivatives of all orders exist and are continuous.

1. a) Let $A(t)$ and $B(t)$ be $n\times n$ matrices whose elements depend smoothly on the real variable t. Use the definition of the derivative (as a limit) to show that their product, $G(t) = A(t)B(t)$, is differentiable.
   What is the derivative of $A^2(t)$?

   b) Give an example of a 2×2 matrix $A(t)$ that depends smoothly on the real variable t with

$$\frac{dA^2(t)}{dt} \neq 2A(t)A'(t).$$

2. Consider two smooth plane curves $\gamma_1, \gamma_2 \colon (0,1) \to \mathbb{R}^2$ that do not intersect. Suppose P_1 and P_2 are interior points on γ_1 and γ_2, respectively, such that the distance $|P_1P_2|$ is minimal. Prove that the straight line P_1P_2 is perpendicular to *both* curves.

3. Let $A(t)$ be a square matrix that depends continuously on t for all $t\in\mathbb{R}$ and let the vector $u(t)$ be a solution of the differential equation

$$\frac{du(t)}{dt} = A(t)u(t) \qquad \text{with} \qquad u(0) = 0.$$

   Show that $u(t)\equiv 0$. [SUGGESTION: Let $E(t) = |u(t)|^2$. You will need the Schwarz inequality and that $|A(t)u(t)| \le |A(t)||u(t)|$.]

4. Let $w(x)$ be a smooth function that satisfies $w'' - c(x)w = 0$, where $c(x) > 0$ is a given continuous function.

   a) Show that w cannot have a local positive maximum (that is, a local maximum where the function is positive). Also show that w cannot have a local negative minimum.

   b) Show that if in addition there are points $a < b$ where $w(a) = 0$ and $w(b) = 0$, then $w(x) = 0$ for all $a \le x \le b$.

   c) Give an example with $c(x) < 0$ where the conclusion of the previous part fails.

5. Let $a(\theta) > 0$ and $b(\theta) > 0$ be positive smooth functions for $\theta\in[0,2\pi]$ and assume they are periodic with period 2π, so, for instance $a(\theta+2\pi) = a(\theta)$ for all real θ. In brief, we are working on a circle. Think of a smooth function $u(\theta)$ as measuring the temperature, so u and

all of its derivatives are periodic with period 2π. Say $u(\theta)$ is a (2π periodic) solution of the nonlinear equation

$$u'' = a(\theta)e^u - b(\theta).$$

a) Find an upper bound for $u(\theta)$ in terms of $a(\theta)$ and $b(\theta)$. [SUGGESTION: Look at the point where u has its maximum value.]

b) Find a lower bound for $u(\theta)$ in terms of $a(\theta)$ and $b(\theta)$.

6. Use the definition of the integral as the limit of a sum to compute

$$\text{a).}\quad \int_0^b x^2\,dx \qquad\qquad \text{b).}\quad \int_0^x \cos\theta\,d\theta.$$

[See `http://www.math.upenn.edu/~kazdan/202F09/sum-sin_kx.pdf`]

7. Compute $\lim_{\lambda\to\infty} \int_0^1 |\sin(\lambda x)|\,dx$.

Bonus Problems (Due Nov 11)

B-1 Let $f(x)$ be a continuous function for $0 \le x \le 1$. Evaluate $\lim_{n\to\infty} \int_0^1 nf(x)x^n\,dx$. (Justify your assertions.)

B-2 For $x > 0$ define the function

$$H(x) = \int_1^x \frac{1}{t}\,dt.$$

Since the integrand, $1/t$ is a continuous function on the interval $[1,x]$ (if $x \ge 1$) or $[x.1]$ (if $x \le 1$), this is Riemann integrable.

Use the definition of the Riemann integral directly to show that for any $y > 0$,

$$H(x) + H(y) = H(xy), \tag{1}$$

thus establishing that $H(x)$ has the basic property of the logarithm.

SUGGESTION: First prove (1) assuming $x \ge 1$ (and any $y > 0$) by rewriting (1) in the form $H(x) = H(xy) - H(y)$, that is,

$$\int_1^x \frac{1}{t}\,dt = \int_y^{xy} \frac{1}{s}\,ds$$

and use a geometric argument that relates a Riemann sum for the integral on the left to a corresponding Riemann sum on the right. [First try the special case $x = 2$, $y = 2$.]

If $0 < x < 1$, then $1/x > 1$, so the result (1) follows from the case $x \geq 1$ by the clever chain:

$$\begin{aligned} H(x) + H(y) &= H(x) + H(\frac{1}{x}xy) = H(x) + [H(\frac{1}{x}) + H(xy)] \\ &= H(\frac{1}{x}) + H(x) + H(xy) = H(\frac{1}{x}x) + H(xy) = H(1) + H(xy) = H(xy). \end{aligned}$$

[Last revised: November 6, 2010]