Math508, Fall 2010 **Jerry L. Kazdan**

Problem Set 6

DUE: Thurs. Oct. 28, 2010. *Late papers will be accepted until 1:00 PM Friday.*

1. Give examples of the following:
   a) An open cover of $\{x \in \mathbb{R} \,:\, 0 < x \leq 1\}$ that has no finite sub-cover.
   b) A metric space having a bounded infinite sequence with no convergent subsequence.
   c) A metric space that is not complete.

2. Let K be a compact set in a metric space $\mathcal{M}$ and let $p \in \mathcal{M}$ be a point *not* in K. Define the distance $\operatorname{dist}(p,K)$ between p and K as
$$\operatorname{dist}(p,K) = \inf_{x \in K} d(p,x).$$
   a) Show there is at least one point $q \in K$ that has this minimum distance, so $d(p,q) = \operatorname{dist}(p,K)$
   b) Is there a *unique* such point q? Proof or counterexample.
   c) Is the assertion in part a) still true if you only assume that K is a closed (but not compact) subset of $\mathbb{R}^2$? Proof or counterexample.

3. For any two sets S, T in a metric space, define the *distance* between these sets as
$$\operatorname{dist}(S,T) = \inf_{x \in S,\, y \in T} d(x,y).$$
   Assume both S and T are compact, and their intersection, $S \cap T$, is empty.
   a) Prove that there are points $p \in S$ and $q \in T$ with $\operatorname{dist}(S,T) = d(p,q)$.
   b) Is $\operatorname{dist}(S,T) > 0$ necessarily true? Justify your assertion.
   c) Give an example of disjoint closed sets S, T in $\mathbb{R}^2$ with the property that $\operatorname{dist}(S,T) = 0$.

4. Let $f : \mathbb{R}^n \to \mathbb{R}^k$ have the property; for some constant m one has
$$|f(x) - f(\hat{x})| \leq m|x - \hat{x}| \quad \text{for all} \quad x, \hat{x} \in \mathbb{R}^n.$$
   Show that f is uniformly continuous on $\mathbb{R}^n$.
   REMARK: We will later show that if f is differentiable and its derivative is bounded by m, then it has the above property. This is one version of the *mean value theorem*.

5. Let $f:\mathbb{R}\to\mathbb{R}$ have the property: $|f(x)-f(\hat{x})|\le c|x-\hat{x}|$ for all real x, $\hat{x}$, where $0\le c<1$ is a constant. Given any starting point x_0, define x_j, $j=1,2,\ldots$, recursively by the rule

$$x_{j+1}=f(x_j).$$

Prove that the x_j converge to some real number, say p, and that $f(p)=p$. In other words, p is a *fixed point* of f.

If $c=1$, give an example of a function f that has no fixed points.

6. Show, directly from the definition, that $\sqrt{x}$ is continuous at every $x\ge 0$. Is it uniformly continuous for every $x\in[0,\infty)$? Why?

7. Which of the following are uniformly continuous in the set $\{x\ge 0\}$? Justify your assertions.

a). $f(x)=2+3x$ b). $g(x)=\sin 2x$ c). $h(x)=1+x^2$ d). $k(x)=\sqrt{x+1}$,

8. Assume that $f(x)$ is uniformly continuous on the bounded open interval $a<x<b$. Prove that f is bounded, that is, there is some M so that $|f(x)|\le M$ for all $x\in (a,b)$.

9. Let $E\subset\mathbb{R}$ be a set and $f:E\to\mathbb{R}$ be uniformly continuous.
   a) If E is a bounded set, show that $f(E)$ is a bounded set.
   b) If E is not bounded, give an example showing that $f(E)$ might not be bounded.
   c) If $f:\mathbb{R}\to\mathbb{R}$ is uniformly continuous on *all* of $\mathbb{R}$, show there are constants a and b so that

$$|f(x)|\le a+b|x|.$$

Bonus Problem (Due Oct 28)

B-1 Let $f(x)$ be a continuous real-valued function with the property

$$f(x+y)=f(x)+f(y)$$

for all real x, y. Show that $f(x)=cx$, where $c:=f(1)$. [Hint: $f(2)=?$]

REMARK: There is a very short proof if you assume f is differentiable.

B-2 Define $f(z)$ for complex z by the power series $f(z):=\sum_{k=0}^{\infty}a_kz^k$,

Assume this series converges in the disk $|z|<R$. Prove (with your bare hands) that f is continuous at every point of this (open) disk. [REMARK: You might (or might not) find it simpler to prove the stronger statement that if $0<r<R$, then $f(z)$ is *uniformly continuous* in the closed disk $\{|z|\le r\}$.]

[Last revised: December 16, 2010]