Math508, Fall 2010 **Jerry L. Kazdan**

Problem Set 5

DUE: Thurs. Oct. 21, 2010. *Late papers will be accepted until 1:00 PM Friday.*

1. [Ratio Test] Let a_k be a sequence of complex numbers. Ley $s := \limsup \left|\frac{a_{k+1}}{a_k}\right|$. By comparison with a geometric series, show that the series $\sum a_k$ converges absolutely if $s < 1$.

2. Let A be a square matrix.

   a) Show that $e^{(s+t)A} = e^{sA}e^{tA}$ for all real or complex s, t.

   b) If $AB = BA$, the Hoffman text (p. 48) shows that $e^{A+B} = e^A e^B$. Give an example showing this may be false if A and B don't commute.

   c) If A is any square matrix, show that e^A is invertable.

   d) If A is a 3×3 diagonal matrix, compute e^A.

   e) If A and B are similar matrices (so $A = S^{-1}BS$ for some invertible matrix S), show that $e^A = S^{-1}e^B S$. [In particular, if A is similar to a diagonal matrix D, then by the previous part, $e^A = S^{-1}e^D S$ is easy to compute.]

   f) If $A^2 = 0$, compute e^A.

   g) Compute e^A for the matrix $A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

   h) If P is a projection (so $P^2 = P$) and $t \in \mathbb{R}$, compute e^{tP}.

   i) If R is a reflection (so $R^2 = I$) and $t \in \mathbb{R}$, compute e^{tR}.

   j) For real t show that
   $$e^{\begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$
   (The matrix on the right is a rotation of $\mathbb{R}^2$ through the angle t).

   k) If a (square) matrix A satisfies $A^2 = \alpha^2 I$, show that
   $$e^A = \cosh\alpha\, I + \frac{\sinh\alpha}{\alpha} A.$$

   l) If a square matrix A satisfies $A^3 = \alpha^2 A$ for some real or complex α, show that
   $$e^A = I + \frac{\sinh\alpha}{\alpha} A + \frac{\cosh\alpha - 1}{\alpha^2} A^2.$$
   [This formula generalizes the previous part.] What is the formula if $A^3 = -\alpha^2 A$?

3. [Hoffman, p.55#4] Let $\{x_k\}$ be a Cauchy sequence in an arbitrary metric space $\mathcal{M}$. If a subsequence converges to some element $A \in \mathcal{M}$, show that the original sequence also converges to A.

4. [Hoffman, p.55 #5] Let $\{x_k\}$ be a bounded sequence of real numbers with the property that $|x_{k+1} - x_k| = 1$ for each $k = 1, 2, \ldots$. Show that this sequence has only a finite number of accumulation points.

5. [Hoffman, p.62 #1-2] The point of this problem is for you to prove with your bare hands that $\mathbb{R}^n$ is connected. As usual, I suggest first trying the special case $n = 1$.

   a) Show that the empty set and all of $\mathbb{R}^n$ are both open and closed sets.

   b) Conversely, if a set $S \subset \mathbb{R}^n$ is both open and closed, show it is either the empty set or all of $\mathbb{R}^n$.

6. [Hoffman, p.62 #4] Which of the following subsets of $\mathbb{C}$ are open? Closed?

   a) All z such that $z = \bar{z}$.

   b) All z that satisfy $z\bar{z} > 2$.

   c) All $z \neq 0$ such that $|z| \leq 1$.

   d) All z such that $|z|$ is a rational number.

7. [Hoffman, p.62 #13] Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^k$ be given sete. The *Cartesian product* $A \times B$ is the set of all points $(x, y) \in \mathbb{R}^{n+k}$ where $x \in A$ and $y \in B$. Prove that

   a) if both A and B are bounded, then so is $A \times B$.

   b) if both A and B are open, then so is $A \times B$;

   c) if both A and B are closed, then so is $A \times B$;

8. Show that the set of all $n \times n$ real orthogonal matrices is both closed and bounded. [REMARK: There are many (equivalent) definitions for an orthogonal matrix. Use whichever you prefer.]

Bonus Problems (Due Oct 21)

1B. In class we defined the normed linear space ℓ_2 as the set of all real (or complex) sequences $x = (x_1, x_2, \ldots)$ such that $\|x\|_2 := \left[\sum |x_j|^2\right]^{1/2} < \infty$. Prove this spaceis complete.
SUGGESTION: As a model, see the example of ℓ_1 in
http://www.math.upenn.edu/~kazdan/508F08/completeness-l_1.pdf

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