Bonus Problem for Set 4

1. Define two real numbers x and y to be equal if $|x-y|$ is an integer. We write $x \equiv y \pmod 1$. Thus we have a "topological circle" whose "circumference" is one.

   Let α be an *irrational* real number, $0<\alpha<1$ and consider its integer multiples, $\alpha,\ 2\alpha,\ 3\alpha \ldots \pmod 1$. Show that this set is dense in $0 \le x \le 1$.

   SOLUTION Given any $\varepsilon > 0$ we'll show that every point in $0 \le x \le 1$ is $\pmod 1$ within ε of an integer multiple of α.

   Pick an integer $K>0$ so that $1/K < \varepsilon$. Partition the interval $[0,1]$ into the K intervals $[0,1/K],\ [1/K,1/2K],\ \ldots,[(K-1)/K,1]$, each of width $1/K<\varepsilon$. Consider the $K+1$ (distinct!) points $\alpha,\ 2\alpha,\ldots(K+1)\alpha,\ \pmod 1$. Since there are $K+1$ points and only K intervals, at least 2 of them must lie in one of the intervals. Say $j\alpha$ and $k\alpha$ are in the same interval. Then $|j\alpha - k\alpha| < 1/K < \varepsilon \pmod 1$.

   CASE 1 $k>j$ Let $\ell = k-j>0$. Then $0<\ell\alpha<\varepsilon \pmod 1$. For some integer N a finite number of the intervals

   $$[0,\ell\alpha],\ [\ell\alpha,2\ell\alpha],\ldots[(N-1)\ell\alpha,N\ell\alpha] \pmod 1.$$

   cover the interval $[0,1]$. Thus every number in $[0,1]$ is within ε of one of the numbers $\ell\alpha$, $\ldots N\ell\alpha$.

   CASE 2 $k<j$. This is essentially identical to CASE 1 – except that since $\ell=k-j<0$, the points $\ell\alpha,\ 2\ell\alpha,\ldots$ run backwards through $[0,1]$.

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