Problem Set 4

DUE: Thurs. Oct 7, 2010. *Late papers will be accepted until 1:00 PM Friday.*

1. a) Calculate $\lim_{n\to\infty} \dfrac{5n+17}{n+2}$.

   b) Let $a_n := \dfrac{3n^2-2n+17}{n^2+21n+2}$. Calculate $\lim_{n\to\infty} a_n$.

2. Investigate the convergence or divergence of $\sum a_n$ if

   a). $a_n = \sqrt{n+1}-\sqrt{n}$ b). $a_n = \dfrac{\sqrt{n+1}-\sqrt{n}}{n}$ c). $a_n = \dfrac{1}{1+z^n}$ (complex z)

3. Let $\{a_n\}$ and $\{b_n\}$ be any real bounded sequences.

   a) Show that
   $$\limsup_{n\to\infty}(a_n+b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$
   provided the sum on the right is not of the form $\infty - \infty$.

   b) Give an explicit example where strict inequality can occur.

4. [Hoffman, p. 36 #10] Let S be a (linear) subspace of $\mathbb{R}^n$. If $X \in \mathbb{R}^n$, let $P(X)$ be the orthogonal projection of X into the subspace S. If X_k converges to X, show that $P(X_k)$ converges to $P(X)$.

5. If $\{b_k\}$ is a sequence of positive numbers, prove the *arithmetic-geometric mean inequality*
   $$[b_1 b_2 \cdots b_n]^{1/n} \le \frac{b_1+\cdots+b_n}{n}.$$
   When does equality hold?

6. Assume $a_n > 0$. If $\sum a_n$ converges and $\{b_n\}$ is bounded, prove that $\sum a_n b_n$ converges.

The next three problems are variations on just one idea.

7. Let $\{a_n\}$ be a sequence of real numbers with the property that
   $$|a_{k+1}-a_k| \le \tfrac{1}{2}|a_k-a_{k-1}|, \qquad k = 1,2,\dots\ .$$
   Show that this sequence converges to some real number.

8. a) Let $X_j, j = 1, 2, \dots$ be a sequence of points in $\mathbb{R}^3$. If $\|X_{j+1} - X_j\| \le \frac{1}{j^4}$, show that these points converge.

   b) Let $\{X_j\}$ be a sequence of points in $\mathbb{R}^n$ with the property that

$$\sum_j \|X_{j+1} - X_j\| < \infty.$$

   Prove that the sequence $\{X_j\}$ converges. Give an example of a convergent sequence that does not have this property.

9. In a metric space M let $d(x,y)$ denote the distance. A sequence x_j is called a *fast Cauchy sequence* if $\sum_j d(x_{j+1}, x_j) < \infty$.

   a) In $\mathbb{R}$ give an example of a fast Cauchy sequence and also of a Cauchy sequence that is *not* fast.

   b) Show that every fast Cauchy sequence is indeed a Cauchy sequence.

   c) If there is a constant $0 < c < 1$ such that for all j

$$d(x_{j+1}, x_j) < c d(x_j, x_{j-1})$$

   show that x_j is a fast Cauchy sequence.

Bonus Problems (Due Oct. 7)

1B Define two real numbers x and y to be equal if $|x - y|$ is an integer, thus we have a "topological circle" whose "circumference" is one.

Let α be an *irrational* real number, $0 < \alpha < 1$ and consider its integer multiples, α, 2α, 3α $\dots$. Show that this set is dense in $0 \le x \le 1$.

[Last revised: December 16, 2010]