Problem Set 11

DUE: Never

Note: We say a function is *smooth* if its derivatives of ball orders exist and are continuous.

1. Partition $[a,b] \in \mathbb{R}$ into sub-intervals $a < x_1 < x_2 < \cdots < x_n = b$. A function $h(x)$ that is constant on each sub-interval is called a *step function*. Show that if $f\mathbb{C}([a,b])$, then it can be approximated arbitrarily closely (in the uniform norm) by a step function.

2. Let $f \in C([-1, 1])$ be an even function (so $f(-x) = f(x)$). Show it can be approximated arbitrarily closely (in the uniform norm) by an even polynomial.

3. Let $f \in C^1([0,2])$. Given any $\varepsilon > 0$ show there is a polynomial $p(x)$ such that
$$\max_{x\in[0,2]} |f(x) - p(x)| + \max_{x\in[0,2]} |f'(x) - p'(x)| < \varepsilon$$
That is, $\|f - p\|_{C^1([0,2])} < \varepsilon$.

4. a) Give an example of a continuous function $f : (0,1] \to (0,1]$ that has *no* fixed points.
   b) Let $A \in (1,3)$ and $f(x) := (x/2) + (A/2x)$. Show that f satisfies the hypotheses of the Contracting Mapping Principle on the domain $[1,\infty)$. What is the fixed point?

5. Let $h(x,y)$ and $f(x)$ be continuous for $0 \le x \le 2$, $0 \le y \le 2$.
   a) Show that if $0 < c \le 2$ is sufficiently small, then their is a continuous function $u(x)$ that satisfies
   $$u(x) = f(x) + \int_0^c h(x,y)u(y)\,dy. \qquad (*)$$
   b) In the special case where $h(x,y) \equiv 1$ and $f(x) \equiv 1$, solve equation (*) explicitly. [This is easy. Let $\alpha = \int_0^1 u(y)\,dy$ and then use (*) to solve for α].
   From this, show that indeed for some value of c a solution may *not* exist.

6. Let $f(x)$ and $h(x,y)$ be as in the previous problem. Show that if $\lambda > 0$ is sufficiently small, the equation
$$u(x) = f(x) + \lambda \int_0^2 h(x,y)u(y)\,dy. \qquad (1)$$
has a unique continuous solution $u(x)$.

7. Let f be an even continuous function on $[-1,1]$ with $\int_{-1}^1 f(x)x^n\,dx = 0$ for all *even* $n \ge 0$. Show that $f \equiv 0$.

[Last revised: December 8, 2010]