Math508, Fall 2010 **Jerry L. Kazdan**

Problem Set 10

DUE: Tues. Nov. 30, 2010. *Late papers will be accepted until 1:00 PM Wednesday.*

Note: We say a function is *smooth* if its derivatives of ball orders exist and are continuous.

1. Find an integer N so thst $\quad 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{N}>100$.

2. Let $c(x)$ be a given smooth function and $u(x)\not\equiv 0$ satisfy the differential equation $-u''+c(x)u=\lambda u$ on the bounded interval $\Omega=\{a<x<b\}$ with $u=0$ on the boundary of Ω. Here λ is a constant. Show that
$$\lambda=\frac{\int_\Omega(u'^2+cu^2)\,dx}{\int_\Omega u^2\,dx}$$

3. The *Gamma function* is defined by $\Gamma(x):=\int_0^\infty e^{-t}t^{x-1}\,dt$.
   a) For which real x does this improper integral converge?
   b) Show that $\Gamma(x+1)=x\Gamma(x)$ and deduce that $\Gamma(n+1)=n!$ for any integer $n\geq 0$.

4. Consider $f(x):=\sum_{k=1}^{\infty}\frac{\sin kx}{1+k^4}$.
   a) For which real x is f continuous?
   b) Is f differentiable? Why?

5. If the complex power series $\sum_{k=0}^{\infty}a_k z^k$ converges at $z=c$, and $R<|c|$, show that it converges absolutely and uniformly in the disk $\{z\in\mathbb{C}\,|\,|z|\leq R\}$.

6. Let a_n be a bounded sequence of complex numbers and
$$f(z)=\sum_{n=1}^{\infty}\frac{a_n}{n^z},$$
where $z=x+iy$. If $c>1$, show that this series converges absolutely and uniformly in the half-plane $\{z=x+iy\in\mathbb{C}\,|\,x\geq c\}$.

7. Show that the sequence of functions $f_n(x):=n^3x^n(1-x)$ does not converge uniformly on $[0,1]$.

8. For each of the following give an example of a sequence of continuous functions. Justify your assertions. [A clear sketch may be adequate — as long as it is convincing].
   a) $f_n(x)$ that converge to zero at every x, $0 \le x \le 1$, but *not* uniformly.
   b) $g_n(x)$ that converge to zero at every x, $0 \le x \le 1$, but $\int_0^1 g_n(x)\,dx \ge 1$.
   c) $h_n(x)$ converge to zero uniformly for $0 \le x < \infty$, but $\int_0^\infty h_n(x)\,dx \ge 1$.

Bonus Problems (Due Nov 30)

B-1 If $\varphi\colon \mathbb{R} \to \mathbb{R}$ is continuous with period P, so $\varphi(x+P) = \varphi(x)$ for all real x. Show that

$$\lim_{\lambda\to\infty} \int_0^1 f(x)\varphi(\lambda x)\,dx = \overline{\varphi} \int_0^1 f(x)\,dx,$$

where $\overline{\varphi} := \frac{1}{P}\int_0^P \varphi(t)\,dt$ is the average of φ over one period. [This generalized both HW8 #7 and HW9 #B-1.]

B-2 Let $\varphi_n(t)$ be a sequence of smooth real-valued functions with the properties

$$(a)\ \varphi_n(t) \ge 0, \qquad (b)\ \varphi_n(t) = 0 \text{ for } |t| \ge 1/n, \qquad (c)\ \int_{-\infty}^{\infty} \varphi_n(t)\,dt = 1.$$

Note: because of (b), this integral is only over $-1/n \le t \le 1/n$.

Assume $f(x)$ is uniformly continuous for all $x \in \mathbb{R}$ and define

$$f_n(x) := \int_{-\infty}^{\infty} f(x-t)\varphi_n(t)\,dt.$$

Show that $f_n(x)$ converges uniformly to $f(x)$ for all $x \in \mathbb{R}$. [SUGGESTION: Use $f(x) = f(x)\left(\int_{-\infty}^{\infty} \varphi_n(t)\,dt\right) = \int_{-\infty}^{\infty} f(x)\varphi_n(t)\,dt$. Also, note *explicitly* where you use the uniform continuity of f].

REMARK: One can show that the approximations f_n are also smooth. Thus, this proves that you can approximate a continuous function *uniformly* on any compact set by a smooth function.

[Last revised: November 21, 2010]