Problem Set 1

DUE: Thurs. Sept. 16, 2010. *Late papers will be accepted until 1:00 PM Friday.*

1. Let $x_0 = 1$ and define $x_k := \sqrt{3x_{k-1}+4}$, $k = 1,2,\ldots$. Show that $x_k < 4$ and that the x_k are increasing.

2. Show that $1+\dfrac{1}{2!}+\dfrac{1}{3!}+\cdots+\dfrac{1}{k!} < 3$.

3. Let $A := (a_{ij})$ be a $n\times n$ matrix of complex numbers with $|a_{ij}| \le M$ and let $a_{ij}^{(k)}$ be the elements of A^k, $k = 1,2,\ldots$. Find an estimate for $|a_{ij}^{(k)}|$ in terms of k, n, and M.

4. a) Let $z,w,v \in \mathbb{C}$ and define $d(z,w)) := \dfrac{|z-w|}{1+|z-w|}$. Show that

$$d(z,v) \le d(z,w)+d(w,v) \qquad \text{[triangle inequality]}.$$

   b) Let S be an arbitrary set with $p,q,r \in S$. Say there is a function $g : S\times S \to \mathbb{R}$ that satisfies the triangle inequality

$$g(p,r) \le g(p,q)+g(q,r).$$

   Define $d(p,q)) := \dfrac{g(p,q)}{1+g(p,q)}$. Show that this function $d(p,q)$ also satisfies the triangle inequality.

5. Suppose $a \in \mathbb{R}^k$, $b \in \mathbb{R}^k$, and $x \in \mathbb{R}^k$. Find all $c \in \mathbb{R}^k$ and $r > 0$ (depending on a and b) such that $|x-a| = 2|x-b|$ is satisfied if and only if $|x-c| = r$.

   As an alternate, you may prefer the following generalization. For real $\lambda > 0$, $\lambda \ne 1$, consider the points $x \in \mathbb{R}^k$ that satisfy

$$|x-a| = \lambda|x-b|.$$

   Show that these points lie on a sphere. Part of this is to find the center and radius of this sphere in terms of a, b and λ. What if $\lambda = 1$?

[Last revised: September 18, 2010]