Completeness of ℓ_1

Let ℓ_1 be the vector space of infinite sequences of real numbers $X = (x_1, x_2, \dots)$ with finite norm $\|X\| := \sum_{j=1}^{\infty} |x_j|$. Here we show this space is complete. The proof is a bit fussy.

STEP 1: FIND A CANDIDATE FOR THE LIMIT. Let $X_n = (x_1^{(n)}, x_2^{(n)}, \dots)$ be a Cauchy sequence. Because

$$|x_1^{(n)} - x_1^{(\ell)}| \le \|X_n - X_\ell\|,$$

the first coordinates $x_1^{(n)}$ are a Cauchy sequence of real numbers and hence converge to some real number z_1. Similarly, the other coordinates converge: $\lim_{n\to\infty} x_j^{(n)} \to z_j$ for some real numbers z_j. We thus suspect that $X_n \to Z := (z_1, z_2, \dots)$.

STEP 2: SHOW THAT THIS CANDIDATE Z IS IN ℓ_1. We show that $\|Z\| = \sum_j |z_j| = \lim_{J\to\infty} \sum_{j=1}^{J} |z_j| < \infty$. Now

$$\sum_{j=1}^{J} |z_j| = \sum_{j=1}^{J} \lim_{n\to\infty} |x_j^{(n)}| = \lim_{n\to\infty} \sum_{j=1}^{J} |x_j^{(n)}|.$$

Note that in the second equality there are no difficulties interchanging a limit with the sum of a *finite* number of real numbers. Since Cauchy sequences are bounded, there is an M such that $\|X_n\| < M$ for all n. Thus, for any J

$$\sum_{j=1}^{J} |x_j^{(n)}| \le \sum_{j=1}^{\infty} |x_j^{(n)}| = \|X_n\| < M.$$

Letting $n \to \infty$ we find that $\sum_{j=1}^{J} |z_j| \le \|X_n\| < M$. Because J is arbitrary, we conclude that $\|Z\| \le M$ and hence $Z \in \ell_1$.

STEP 3: PROVE THE CONVERGENCE. Let Finally we show that $\|X_n - Z\| \to 0$. Given $\varepsilon > 0$ pick N so that if n and $\ell > N$ then $\|X_n - X_\ell\| < \varepsilon$. Consequently, for any fixed J and $n, \ell > N$, we find

$$\sum_{j=1}^{J} |x_j^{(n)} - x_j^{(\ell)}| \le \sum_{j=1}^{\infty} |x_j^{(n)} - x_j^{(\ell)}| = \|X_n - X_\ell\| < \varepsilon.$$

With $n > N$ and J fixed, let $\ell \to \infty$ to find

$$\sum_{j=1}^{J} |x_j^{(n)} - z_j| = \lim_{\ell\to\infty} \sum_{j=1}^{J} |x_j^{(n)} - x_j^{(\ell)}| \le \varepsilon.$$

But now this is true for all J so $\|X_n - Z\| \le \varepsilon$, as desired.

Exercise Let ℓ_2 be the vector space of infinite sequences of real numbers $X = (x_1, x_2, \dots)$ with finite norm $\|X\| := \sqrt{\sum_{j=1}^{\infty} |x_j|^2}$. Show this space is complete. [This is Hilbert's *Hilbert Space*]