Math 508, Fall 2008 **Jerry Kazdan**

Homework Set 6

DUE: Thurs. Oct. 30, 2008. Late papers accepted until 1:00 Friday.

1. Let $\mathcal{M}$ and $\mathcal{N}$ be metric spaces and $f: \mathcal{M} \to \mathcal{N}$ be a continuous map. Say $f: p \mapsto q$ and $r \in \mathcal{N}$ with $r \neq q$. Show there is some neighborhood of p whose image does not contain r. In other words, there is some open set $U \subset \mathcal{M}$ containing p with the property that $r \notin f(U)$.

2. Let f be a continuous map from $[0, 1]$ to itself. Show that f has at least one *fixed point*, that is, a point c so that $f(c) = c$.

3. Show that at any time there are at least two diamentically opposite points on the equator of the earth with the same temperature.

4. [Rudin, p. 98 # 3]. Let $\mathcal{M}$ be a metric space and $f: \mathcal{M} \to \mathbb{R}$ a continuous function. Denote by $Z(f)$ the *zero set of* f. These are the points $p \in \mathcal{M}$ where f is zero, $f(p) = 0$.
   a) Show that $Z(f)$ is a closed set.
   b) [See also Rudin, p. 101 #20] Given *any* set $E \in \mathcal{M}$, the distance of a point x to E is defined by
   $$h(x) = \rho_E(x) := \inf_{z \in E} d(x, z).$$
   Show that h is a uniformly continuous function.
   c) Use the previous part to show that given any *closed* set $E \in \mathcal{M}$, there is a continuous function that is zero on E and positive elsewhere.

5. [Rudin, p. 99 # 7]. For points $(x, y) \neq (0, 0) \in \mathbb{R}^2$, define
   $$f(x, y) = \frac{xy^2}{x^2 + y^4} \quad \text{and} \quad g(x, y) = \frac{xy^2}{x^2 + y^6},$$
   while define $f(0, 0) = 0$ and $g(0, 0) = 0$.
   a) Show that f is bounded in $\mathbb{R}^2$ but not continuous at the origin, while g is unbounded in every neighborhood of the origin and hence also not continuous there.

b) Let $S \in \mathbb{R}^2$ be any straight line through the origin. Show that if the points (x,y) are stricted to lie on S, then both $f(x,y)$ and $g(x,y)$ are continuous. MORAL: It can be misleading to understand a function by only examining it on straight lines.

6. Let $f(x)$ be a continuous real-valued function with the property

$$f(x+y) = f(x) + f(y)$$

for all real x, y. Show that $f(x) = cx$ for some constant c. [REMARK: There is a very short proof if you assume f is differentiable].

7. [Partly from Rudin, p. 99 # 8]. Let $E \subset \mathbb{R}$ be a set and $f : E \to \mathbb{R}$ be uniformly continuous.

   a) If E is a bounded set, show that $f(E)$ is a bounded set.

   b) If E is not bounded, give an example showing that $f(E)$ might not be bounded.

   c) If $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on *all* of $\mathbb{R}$, show there are constants a and b so that

$$|f(x)| \leq a + b|x|.$$

8. Define $f(z)$ for complex z by the power series $f(z) := \sum_{k=0}^{\infty} a_k z^k$,

   Assume this series converges in the disk $|z| < R$. Prove that f is continuous at every point of this (open) disk.

9. [Rudin, p. 99 # 13 or #11, see also p. 98 #4] *extension by continuity* Let X be a metric space, $E \subset X$ a dense subset, and $f : E \to \mathbb{R}$ a uniformly continuous function. Show that f has a unique continuous extension to all of X. That is, there is a unique continuous function $g : X \to \mathbb{R}$ with the property that $g(p) = f(p)$ for all $p \in X$. [REMARK: One generalize this by replacing $\mathbb{R}$ by any complete metric space.]