Homework Set 5

DUE: Thurs. Oct. 23, 2008. Late papers accepted until 1:00 Friday.

1. a) Let A be a subset of the real numbers. Prove that the following statements are equivalent:
   - A is closed.
   - Every sequence $x_n \in A$ such that $\sum_{n=1}^{\infty} \|x_{n+1} - x_n\| < \infty$ converges to a limit in A.

   b) Show this is also true for subsets A of the plane $\mathbb{R}^2$ with the usual euclidian norm.

2. Let ℓ_2 be the usual normed linear space space of infinite sequences sequences $X = (x_1, x_2, \ldots)$ with finite norm: $\|X\| := \sqrt{\sum_{j=1}^{\infty} |x_j|^2} < \infty$. Prove that ℓ_2 is *complete*. [SUGGESTION: See the similar proof that ℓ_1 is complete on the class web page.]

3. Let V and W be normed linear spaces and $L : V \to W$ a linear map, so $L(X+Y) = LX + LY$ and $L(cX) = cLX$. Define the *norm* of L by

$$\|L\| := \sup_{X \neq 0} \frac{\|LX\|_W}{\|X\|_V}.$$

   We say that L is *bounded* if $\|L\| < \infty$

   a) The set $\mathcal{L}(V,W)$ of all linear maps from V to W is itself a linear space — since one can add maps, $L+M$, and multiply them by scalars, cL. Show that $\|L\|$ defines a norm on $\mathcal{L}(V,W)$, that is,

      i). $\|L\| \geq 0$, with $\|L\| = 0$ only if $L = 0$,

      ii). $\|cL\| = |c|\|L\|$ for any scalar c,

      ii). $\|L+M\| \leq \|L\| + \|M\|$ (triangle inequality).

   b) Show that $\|L\| = \sup_{\|X\|_V = 1} \|LX\|_W$.

   c) Let $A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 2 & 0 \end{pmatrix}$ define a linear map from $\mathbb{R}^3$ to $\mathbb{R}^2$ with the usual euclidean norms. Show that A is bounded. (You need not compute $\|A\|$, only get an upper bound for it).

   d) If $L : \mathbb{R}^k \to \mathbb{R}^n$ is given by any $n \times k$ matrix and both $\mathbb{R}^k$ and R^n have the Euclidean norm, show that L is bounded.

   e) Show that if $L : \ell_2 \to \ell_2$ is defined by

$$LX := (c_1 x_1, c_2 x_2, c_3 x_3, \ldots),$$

   where c_j is a bounded sequence of complex numbers, then L is bounded.

f) Show that if $L : \ell_2 \to \ell_2$ is defined by

$$LX := (x_1, 2x_2, \ldots, nx_n, \ldots)$$

is *not* a bounded linear map.

4. [CONTINUATION] Show that a linear map $L : V \to W$ is continuous at any point X_0 if and only if L is continuous at the origin.

5. [CONTINUATION] Show that a linear map $L : V \to W$ is continuous if and only if it is bounded.

6. Let $f : \mathbb{R}^n \to \mathbb{R}^k$ have the property that for some constant m one has

$$\|f(x) - f(\hat{x})\| \le m\|x - \hat{x}\| \quad \text{for all} \quad x, \hat{x} \in \mathbb{R}^n.$$

Show that f is uniformly continuous on $\mathbb{R}^n$.
REMARK: We will later show that if f is differentiable and its derivative is bounded by m, then it has the above property. This is one version of the *mean value theorem*.

7. Let $f : \mathbb{R} \to \mathbb{R}$ have the property that $|f(x) - f(\hat{x})| \le c|x - \hat{x}|$ for all real x, $\hat{x}$, and where $0 \le c < 1$ is a constant. Given any starting point x_0, define x_j, $j = 1, 2, \ldots$, recursively by the rule

$$x_{j+1} = f(x_j).$$

Prove that the x_j converge to some real number, say z, and that $f(z) = z$. In other words, z is a *fixed point* of f.