Math 508, Fall 2006 **Jerry L. Kazdan**

Problem Set 2

DUE: Thurs. Sept. 18, 2008. Late papers accepted until 1:00 Friday.

1. (Rudin, p. 23 #12) Let z, w, $z_1, \ldots, z_n$ be complex numbers
   a) Prove the *triangle inequality*: $|z_1+\cdots+z_n| \le |z_1|+\cdots+|z_n|$ (first check for $n=2$).
   b) Show that $||z|-|w|| \le |z-w|$.

2. Let $\mathbb{C}^k$ be the set of all ordered k-tuples $\mathbf{z}=(z_1,\ldots,z_k)$ of complex numbers with the usual definitions of addition and multiplication by (complex) scalars. Define the inner product as
$$\langle \mathbf{z}, \mathbf{w}\rangle = z_1\bar{w}_1+\cdots+z_k\bar{w}_k,$$
and $|\mathbf{z}| = +\sqrt{\langle \mathbf{z}, \mathbf{z}\rangle}$. Prove that Rudin, Theorem 1.37 (page 16) is valid here too.

3. Let a and b be two real numbers with the property that $|b-a|<\varepsilon$ for any rational $\varepsilon>0$. Prove that $a=b$.

4. Consider the set $\mathcal{R}$ of rational functions $f(x)=\dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials with real coefficients and $q(x)$ not identically zero. The function $f(x)$ has a finite value everywhere except at a finite numnber of points (the zeroes of $q(x)$).
   a) It should be obvious that with the usual definitions of addition and multiplication, the set of rational functions is closed under addition and multiplication, that is, the sum and product of two rational functions ia also a rational function. Show that $\mathcal{R}$ forms a field.
   b) In $\mathcal{R}$, define the order $f>0$ to mean that $f(x)>0$ for all sufficiently large positive real x. Thus, if
$$f(x)=\frac{a_0+a_1x+\cdots+a_kx^k}{b_0+b_1x+\cdots+b_nx^n}$$
   with $b_n\neq 0$, then $f>0$ means $\dfrac{a_k}{b_n}>0$. [This gives an algebraic definition of $f>0$ that avoids defining "sufficiently large". Then $f>g$ is defined to mean $f-g>0$. Show that with this order relation, $\mathcal{R}$ is an ordered field.
   c) Show that this ordered field is *non-archemedean* by exhibiting two specific rational functions f and g with the property that there is *no* integer N such that $Nf>g$.

5. (Rudin, p.43 #5) Construct a bounded set of real numbers with exactly three limit points.

6. (Rudin, p.43 #6) Let E' be the set of limit points of a set E in a metric space. Show that E' is closed.

7. (Rudin, p.43 #10) Let X be any set with an infinite number of p[oints. For $p,\ q \in X$ define the function

$$d(p,q) = \begin{cases} 1 & \text{if } p \neq q, \\ 0 & \text{if } p = q \end{cases}$$

Prove that this is a metric (although it is not very interesting). Which subsets are open? closed? compact?

8. (Rudin, p.44 #11) If x and y are real numbers, define
$d_1(x,y) = (x-y)^2$; $\quad d_2(x,y) = \sqrt{|x-y|}$; $\quad d_3(x,y) = |x^2 - y^2|$;
$d_4 = |x - 2y|$; $\quad d_5 = \frac{|x-y|}{1+|x-y|}$.
Which of these define metrics? Justify your assertions.

9. (Rudin, p.45 #22) A metric space is called *separable* if it contains a countable dense subset. Show that $\mathbb{R}^2$ is separable. [HINT: Consider the set of points whose coordinates are rational.]

10. Define two real numbers x and y to be equal if $|x-y|$ is an integer, thus we have a "topological circle" whose circumference is one.

    Let α be an irrational real number, $0 < \alpha < 1$ and consider its integer multiples, α, 2α, 3α .... Show that this set is dense in $0 \leq x \leq 1$.

[Last revised: September 17, 2008]