

Completeness of ℓ_1

Let ℓ_1 be the vector space of infinite sequences of real numbers $X = (x_1, x_2, \dots)$ with finite norm $\|X\| := \sum_{j=1}^{\infty} |x_j|$. Here we show this space is complete.

STEP 1: FIND A CANDIDATE FOR THE LIMIT. Let $X_n = (x_1^{(n)}, x_2^{(n)}, \dots)$ be a Cauchy sequence. Because

$$|x_1^{(n)} - x_1^{(\ell)}| \leq \|X_n - X_\ell\|,$$

the first coordinates $x_1^{(n)}$ are a Cauchy sequence of real numbers and hence converge to some real number z_1 . Similarly, the other coordinates converge: $x_j^{(n)} \rightarrow z_j$ for some real numbers z_j . We thus suspect that $X_n \rightarrow Z := (z_1, z_2, \dots)$.

STEP 2: SHOW THAT THIS CANDIDATE Z IS IN ℓ_1 . We show that $\|Z\| = \sum_j |z_j| < \infty$. Now $\|Z\|$ really means the double limit:

$$\|Z\| := \lim_{J \rightarrow \infty} \sum_{j=1}^J |z_j| = \lim_{J \rightarrow \infty} \left(\lim_{n \rightarrow \infty} \sum_{j=1}^J |x_j^{(n)}| \right).$$

Note that there are no difficulties interchanging a limit with the sum of a finite number of real numbers). But for some M we know that $\|X_n\| < M$ for all n . Thus, for any J

$$\sum_{j=1}^J |x_j^{(n)}| \leq \sum_{j=1}^{\infty} |x_j^{(n)}| = \|X_n\| < M.$$

Letting $n \rightarrow \infty$ we find that $\sum_{j=1}^J |z_j| \leq \|X_n\| < M$. Because J is arbitrary, we conclude that $\|Z\| \leq M$ and hence $Z \in \ell_1$.

STEP 3: PROVE THE CONVERGENCE. Let Finally we show that $\|X_n - Z\| \rightarrow 0$. Given $\varepsilon > 0$ pick N so that if n and $\ell > N$ then $\|X_n - X_\ell\| < \varepsilon$. Consequently, for any fixed J and $n, \ell > N$, we find

$$\sum_{j=1}^J |x_j^{(n)} - x_j^{(\ell)}| \leq \sum_{j=1}^{\infty} |x_j^{(n)} - x_j^{(\ell)}| = \|X_n - X_\ell\| < \varepsilon.$$

With $n > N$ and J fixed, let $\ell \rightarrow \infty$ to find

$$\sum_{j=1}^J |x_j^{(n)} - z_j| = \lim_{\ell \rightarrow \infty} \sum_{j=1}^J |x_j^{(n)} - x_j^{(\ell)}| \leq \varepsilon.$$

But now this is true for all J so $\|X_n - Z\| \leq \varepsilon$, as desired.

Exercise Let ℓ_2 be the vector space of infinite sequences of real numbers $X = (x_1, x_2, \dots)$ with finite norm $\|X\| := \sqrt{\sum_{j=1}^{\infty} |x_j|^2}$. Show this space is complete. [This is Hilbert's *Hilbert Space*]