Math 508 | Exam 1 | Jerry L. Kazdan

October 16, 2008 | | 10:30 – 11:50

DIRECTIONS This exam has three parts, Part A has 4 problems asking for Examples (20 points, 5 points each), Part B asks you to describe some sets (20 points), Part C has 4 traditional problems (60 points, 15 points each).
Closed book, no calculators or computers– but you may use one $3'' \times 5''$ card with notes on both sides.

Part A: Examples (4 problems, 5 points each). Give an example having the specified property.

A–1. A metric space that contains all but one of its limit points.

A–2. An open cover of $\{x \in \mathbb{R} \ : \ 0 < x \le 1\}$ that has no finite sub-cover.

A–3. A metric space having a bounded infinite sequence with no convergent subsequence.

A–4. A metric space that is not complete.

Part B: Classify sets (20 points) For each of the following sets, **circle** the listed properties it has:

a) $\{1 - \frac{1}{n} \in \mathbb{R},\ n = 1, 2, 3, \ldots\}$ open closed bounded compact countable

b) $\{1\} \cup \{1 + \frac{(-1)^n}{n} \in \mathbb{R},\ n = 1, 2, 3, \ldots\}$

open closed bounded compact countable

c) $\{(x, y) \in \mathbb{R}^2 : 0 \le y - x \le 1\}$ open closed bounded compact countable

d) $\{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2\}$ open closed bounded compact countable

e) $\{(x, y) \in \mathbb{R}^2 : x > 1,\ y < \frac{1}{x}\}$ open closed bounded compact countable

f) $\{(k, n) \in \mathbb{R}^2 : k, n$ any positive integers with $k^2 + n^2 < 100\}$

open closed bounded compact countable

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Part C: Traditional Problems (4 problems, 15 points each)

C–1. In $\mathbb{R}$, if $a_n \to A$ and $b_n \to B$, show that the product $a_n b_n \to AB$.

C–2. Given a complex sequence $\{a_k\}$, let $S_n = \dfrac{a_1 + \cdots + a_n}{n}$ be the sequence of averages (*arithmetic mean*). If a_k converges to 0, show that the averages S_n also converge to 0.

C–3. Let $\{a_n\} \in \mathbb{C}$ be a bounded sequence. If $x > 1$ show that $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n^x}$ converges absolutely.

C–4. For any two sets S, T in a metric space, define the *distance* between these sets as

$$\operatorname{dist}(S,T) = \inf_{x \in S,\, y \in T} d(x,y).$$

Assume both S and T are compact, and their intersection, $S \cap T$, is empty.

a) Prove that there are points $p \in S$ and $q \in T$ with $\operatorname{dist}(S,T) = d(p,q)$.

b) Is $\operatorname{dist}(S,T) > 0$ necessarily true? Justify your assertion.