Homework Set 9

DUE: Thurs. Nov. 28, 2006. Late papers accepted until 1:00 Friday.

Math 508, Fall 2006 **Jerry Kazdan**

Note: We say a function is *smooth* if its derivatives of all orders exist and are continuous.

1. Let $\alpha(t)$ and $\beta(s)$ describe smooth curves in $\mathbb{R}^3$ that do not intersect. Say the points $p = \alpha(t_0)$ and $q = \beta(s_0)$ minimize the distance between the curves. Show that the line from p to q is perpendicular to both of these curves.

2. a) Let $y = f(x)$ define a smooth curve in the plane. If P, Q and R are three distinct points on the curve, let Γ_{PQR} be the circle that passes through these three points (we allow that Γ_{PQR} might be a straight line, which can be viewed as a circle with infinite radius). In the limit as both $Q \to P$ and $R \to P$ show that this circle Γ_P is tangent to the curve at P and that in addition the second derivative of the curve and the circle agree at P. [If the circle Γ_P has radius R, we say that the *curvature* of $y = f(x)$ at P is $1/R$.]

   b) Use this to obtain a formula for the curvature in terms of f, f', and f''.

3. Let $f(x) \in C([a,b])$. Show that

$$\exp\left[\frac{1}{b-a}\int_a^b f(x)\,dx\right] \le \frac{1}{b-a}\int_a^b \exp[f(x)]\,dx$$

   [HINT: Use the inequality $e^u \ge 1+u$ where $u = f - \bar{f}$. Here $\bar{f}$ = average of f = $\frac{1}{b-a}\int_a^b f(x)\,dx$.[

4. In number theory, the function $\mathrm{Li}\,(x) := \displaystyle\int_2^x \frac{dt}{\log t}$ arises in estimating the number of primes less than x. Show that $\mathrm{Li}\,(x)$ is asymptotically equal to $x/\log x$ for large x, that is

$$\lim_{x\to\infty} \frac{\mathrm{Li}\,(x)}{\dfrac{x}{\log x}} = 1,$$

5. If $f(x) > 0$ is continuous for all $x \ge 0$ and the limit $\lim\limits_{c\to\infty} \int_0^c f(x)\,dx$ exists, must it be true that $\lim_{x\to\infty} f(x) = 0$? Proof or counterexample.

6. a) If a smooth function $u(x,y)$ satisfies

$$\frac{\partial u}{\partial y} = 0 \quad \text{on all of } \mathbb{R}^2 \text{ and } \quad u(x,0) = 7 + x + \sin 2x,$$

what can you conclude? Why?

b) If a smooth function $v(x,y)$ satisfies

$$\frac{\partial v}{\partial x} - 2\frac{\partial v}{\partial y} = 0 \quad \text{on all of } \mathbb{R}^2 \text{ and } \quad v(x,0) = 7 + x + \sin 2x,$$

what can you conclude? Why?

Some Review Problems

7. Consider the linear space S of real sequences $x = (x_1, x_2, \ldots)$ with only a finite number of non-zero terms. Let $\|x\| := \max_j |x_j|$.

   a) Show that this is a norm on this space.

   b) Is this space complete with this norm? Justify your response.

8. Let X be any metric space and $\mathcal{B}(X,\mathbb{R})$ the metric space of all *bounded* real valued functions $f : X \to \mathbb{R}$ with the metric

$$\rho(f,g) := \sup_{x \in X} |f(x) - g(x)|.$$

Show that this metric space is complete. [First try the case where X is the real interval $0 \le x \le 1$].

9. The nth Legendre polynomial is $P_n(x) = \dfrac{d^n}{dx^n}(x^2 - 1)^n$.

   a) Show that $P_n(x)$ is a polynomial of degree n.

   b) Show that $P_n(x)$ has exactly n real distinct zeroes in the interval $\{-1 < x < 1\}$.

   c) If $k \neq n$, show that $\int_{-1}^{1} P_n(x) P_k(x)\, dx = 0$, in other words, these polynomials are orthogonal in the inner product of $L^2([-1,1])$.