Homework Set 8

DUE: Thurs. Nov. 16, 2006. Late papers accepted until 1:00 Friday.

Math 508, Fall 2006 **Jerry Kazdan**

Note: We say a function is *smooth* if its derivatives of all orders exist and are continuous.

1. Let $\mathbf{r}(t)$ describe a smooth curve in $\mathbb{R}^3$ and let $\mathbf{V}$ be a fixed vector. If $\mathbf{r}'(t)$ is perpendicular to $\mathbf{V}$ for all t and if $\mathbf{r}(0)$ is perpendicular to $\mathbf{V}$, show that $\mathbf{r}(t)$ is perpendicular to $\mathbf{V}$ for all t.

2. A *diffeomorphism* is a smooth invertable map whose inverse map is also smooth.
   a) Find a diffeomorphism $f : \mathbb{R} \to \mathbb{R}_+$, where $\mathbb{R}_+$ is the half-line: $\mathbb{R}_+ = \{x \in \mathbb{R} \,|\, x > 0\}$.
   b) Find a diffeomorphism $g : \mathbb{R} \to \Omega$, where Ω is the interval: $\Omega = \{x \in \mathbb{R} \,|\, 0 < x < 1\}$.
   c) Find a diffeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2_+$, where $\mathbb{R}^2_+$ is the upper half-plane: $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 \,|\, y > 0\}$.
   d) Find a diffeomorphism $G : \mathbb{R}^2 \to \Omega$, where Ω is the strip: $\Omega = \{(x,y) \in \mathbb{R}^2 \,|\, 0 < y < 1\}$.

3. Let $f(x)$ be a smooth function for $x \geq 1$ with the property that $f'(x) \to 0$ as $x \to \infty$.
   a) Show that $f(n+1) - f(n) \to 0$ as $n \to \infty$.
   b) Compute $\lim_{n\to\infty} \sqrt[5]{n+1} - \sqrt[5]{n}$.

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4. For x in any finite interval $|x| \leq c$ prove that $\lim_{N\to\infty} \sum_{k=0}^{N} \frac{x^k}{k!} = e^x$ by showing that the remainder in the Taylor series goes to zero.

5. [ERROR IN INTERPOLATION] Let $f : [a, b] \to \mathbb{R}$ be a smooth function.
   a) Let $g(x)$ be the straight line with the property that $g(a) = f(a)$ and $g(b) = f(b)$. For any point $c \in [a, b]$ obtain an estimate for the error: $f(c) - g(c)$.
   REMARK: Your estimate will involve $f''(z)$ for some point $z \in [a, b]$. The estimate is related to the procedure used to find the error in a Taylor polynomial.

HINT: Define the constant M by $f(c) = g(c) + M(c-a)(c-b)$. Then consider the function

$$\phi(x) := f(x) - g(x) - M(x-a)(x-b).$$

b) Let $a = x_0 < x_2 < \cdots < x_k = b$ and let $g(x)$ be the polynomial of degree k that agrees with f at these $k+1$ points, so $g(x_j) = f(x_j)$, $j = 0, 1, \ldots, k$. Obtain an estimate for the error, $f(c) - g(c)$, for any $c \in [a, b]$

6. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function.

   a) If $f(x) \geq 0$ and $\int_0^1 f(x)\,dx = 0$, prove that $f(x) = 0$ for all $x \in [0, 1]$.

   b) If $\int_0^1 f(x)\,dx = 0$, prove that $f(c) = 0$ for some $c \in (0, 1)$. Even more, prove that $f(x)$ changes sign somewhere in this interval.

   c) If $f : [0, 1] \to \mathbb{R}$ is a continuous function with the property that $\int_0^1 f(x)g(x)\,dx = 0$ for *all* continuous functions g prove that $f(x) = 0$ for all $x \in [0, 1]$.

   d) If $f : [0, 1] \to \mathbb{R}$ is a continuous function with the property that $\int_0^1 f(x)g(x)\,dx = 0$ for *all* C^1 functions g that satisfy $g(0) = g(1) = 0$, must it be true that $f(x) = 0$ for all $x \in [0, 1]$? Proof or counterexample.

7. Let $f(t)$ be a continuous function for $0 \leq t < \infty$. If $\lim_{t\to\infty} f(t) = c$, show that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(t)\,dt = c.$$

8. Let $f(x)$ be a continuous function for $0 \leq x \leq 1$. Evaluate $\lim_{n\to\infty} n \int_0^1 f(x)x^n\,dx$. (Justify your assertions.)

9. a) If $V = (x, y, z) \in \mathbb{R}^3$ and $p \geq 1$, define $\|V\|_p := [|x|^p + |y|^p + |z|^p]^{1/p}$. Show that $\lim_{p\to\infty} \|V\|_p = \max\{|x|, |y|, |z|\}$.

   b) If $f \in \mathbb{C}[0, 2]$ and $p \geq 1$, define

$$\|f\|_p := \left[\int_0^2 |f(x)|^p\,dx\right]^{1/p}.$$

   Show that $\lim_{p\to\infty} \|f\|_p = \max_{0\leq x\leq 2} |f(x|$.

10. Compute $\lim_{\lambda\to\infty} \int_0^1 |\sin(\lambda x)|\,dx$.

11. Let $p(x)$ be a real polynomial of degree n. The following uses the inner product $\langle f, g\rangle := \int_0^1 f(x)g(x)\,dx$.

a) If p is orthogonal to the constants, show that p has at least one real zero in the interval $\{0 < x < 1\}$.

b) If p is orthogonal to all polynomials of degree at most one, show that p has at least two distinct real zeros in the interval $\{0 < x < 1\}$.

c) If p is orthogonal to all polynomials of degree at most $n-1$, show that p has exactly n distinct real zeros in the interval $\{0 < x < 1\}$.

BONUS PROBLEMS

These are more challenging. If you do any of these, please give your solutions directly to me by Thursday, Nov. 30.

Bonus Problem 1 Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function.

a) Show that $\displaystyle\lim_{\lambda\to\infty} \int_0^1 f(x)\sin(\lambda x)\,dx = 0.$

b) (generalization) If $\varphi\colon \mathbb{R} \to \mathbb{R}$ is continuous with period P, show that

$$\lim_{\lambda\to\infty} \int_0^1 f(x)\varphi(\lambda x)\,dx = \overline{\varphi} \int_0^1 f(x)\,dx,$$

where $\overline{\varphi} := \frac{1}{P}\int_0^P \varphi(t)\,dt$ is the average of φ over one period.

Bonus Problem 2 Let $\mathcal{C}$ be the ring of continuous functions on the interval $0 \le x \le 1$.

a) If $0 \le c \le 1$, show that the subset $\{\, f \in \mathcal{C} \mid f(c) = 0\}$ is a maximal ideal.

b) Show that every maximal ideal in $\mathcal{C}$ has this form.

Bonus Problem 3 Let $a_0, a_1, \ldots$ be *any* sequence of real numbers. Show there is a smooth function $f(x)$ with the property that a_n is its n^{th} Taylor coefficient: $a_n = \frac{1}{n!} f^{(n)}(x)\big|_{x=0}$.