Homework Set 7

DUE: Thurs. Nov. 9, 2006. Late papers accepted until 1:00 Friday.

Math 508, Fall 2006 **Jerry Kazdan**

Note: We say a function is *smooth* if its derivatives of all orders exist and are continuous.

1. Which of the following are uniformly continuous in the set $\{x \ge 0\}$? Justify your assertions.

   a). $f(x) = 2 + 3x$ b). $g(x) = \sin 2x$ c). $h(x) = x^2$ d). $k(x) = \sqrt{x}$,

2. a) Show that $\sin x$ is not a polynomial.

   b) Show that $\sin x$ is not a rational function, that is, it cannot be the quotient of two polynomials.

   c) Let $f(t)$ be periodic with period 1, so $f(t+1) = f(t)$ for all real t. If f is not a constant, show that it cannot be a rational function. that is, f cannot be the quotient of two polynomials.

   d) Show that e^x is not a rational function.

3. a) If a smooth curve $y = f(x)$ has the property that $f''(x) \ge 0$, show that it is convex.

   b) Let $v(x)$ be a smooth real-valued function for $0 \le x \le 1$. If $v(0) = v(1) = 0$ and $v''(x) > 0$ for all $0 \le x \le 1$, show that $v(x) \le 0$ for all $0 \le x \le 1$.

   c) Prove that the function e^x is convex.

   d) Show that $e^x \ge 1 + x$ for all real x.

4. a) Let $p(x) := x^3 + cx + d$, where c, and d are real. Under what conditions on c and d does this has three distinct real roots? [ANSWER: $c < 0$ and $d^2 < -4c^3/27$].

   b) Generalize to the real polynomial $p(x) := ax^3 + bx^2 + cx + d$ $(a \ne 0)$ by a change of variable $t = x - \alpha$ (with a clever choice of α) to reduce to the above special case.

5. Let a smooth function $g(x)$ have the three properties: $g(0) = 2$ $g(1) = 0$ $g(4) = 6$. Show that at some point $0 < c < 4$ one has $g''(c) > 0$. Better yet, find a number $m > 0$ so that $g''(c) \ge m > 0$.

   Is it true that g'' must be positive at at least one point $0 < c < 1$? Proof or counterexample.

6. Let $\mathbf{r}(t)$ define a smooth curve that does not pass through the origin.
   a) If the point $\mathbf{a} = \mathbf{r}(t_0)$ is a point on the curve that is closest to the origin (and *not* an end point of the curve), show that the position vector $\mathbf{r}(t_0)$ is perpendicular to the tangent vector $\mathbf{r}'(t_0)$.
   b) What can you say about a point $\mathbf{b} = \mathbf{r}(t_1)$ that is *furthest* from the origin?

7. If $h : \mathbb{R} \to \mathbb{R}$ is a differentiable function that satisfies $h'(t) \le ch(t)$, where c is a constant, show that $h(t) \le e^{ct} h(0)$ for all $t \ge 0$.

8. Say $u(t)$ satisfies $u'' + b(t)u' + c(t)u = 0$, where $b(t)$ and $c(t)$ are bounded functions. Let $E(t) := \frac{1}{2}(u'^2 + u^2)$.
   a) Show that $E'(t) \le \gamma E(t)$, where γ is a constant.
   b) Use the result of the previous problem to deduce that if $u(0) = 0$ and $u'(0) = 0$, then $u(t) = 0$ for all t.

9. Let $w(x)$ be a smooth function that satisfies $w'' - c(x)w = 0$, where $c(x) > 0$ is a given function, show that w cannot have a local positive maximum (that is, a local maximum where the function is positive). Also show that w cannot have a local negative minimum.

10. a) For any integer $n \ge 0$, show that $\displaystyle\lim_{x \searrow 0} \frac{e^{-1/x}}{x^n} = 0$.
    b) Define $f : \mathbb{R} \to \mathbb{R}$ by
    $$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{for } x > 0, \\ 0 & \text{for } x \le 0, \end{cases}.$$
    Sketch the graph of f.
    c) Show that f is a smooth function for all real x
    d) Show that each of the following are smooth and sketch their graphs:
    $$g(x) = f(x)f(1-x) \qquad h(x) = \frac{f(x)}{f(x) + f(1-x)}$$
    $$k(x) = h(x)h(4-x) \qquad K(x) = k(x+2),$$
    $$\varphi(x,y) = K(x)K(y),\ (x,y) \in \mathbb{R}^2 \qquad \Phi(x) = K(\|x\|),\ x = (x_1, x_2) \in \mathbb{R}^2$$