Homework Set 6

DUE: Thurs. Nov. 2, 2006. Late papers accepted until 1:00 Friday.

Math 508, Fall 2006 **Jerry Kazdan**

1. If $L : \ell_2 \to \ell_2$ is defined by $LX := (c_1x_1, c_2x_2, c_3x_3, \ldots)$, where c_j is a bounded sequence, is L is bounded? Proof or counterexample.

2. Show that a linear map $L : V \to W$ between normed vector spaces V and W is continuous at any point X_0 if and only if L is continuous at the origin.

3. [CONTINUATION] Show that a linear map $L : V \to W$ is continuous if and only if it is bounded.

4. Let $\mathcal{M}$ and $\mathcal{N}$ be metric spaces and $f : \mathcal{M} \to \mathcal{N}$ be a continuous map. Say $f : p \mapsto q$ and $r \in \mathcal{N}$ with $r \neq q$. Show there is some neighborhood of p whose image does not contain r. In other words, there is some open set $U \subset \mathcal{M}$ containing p with the property that $r \notin f(U)$.

5. Let f be a continuous map from $[0, 1]$ to itself. Show that f has at least one *fixed point*, that is, a point c so that $f(c) = c$.

6. Show that at any time there are at least two diamentically opposite points on the equator of the earth with the same temperature.

7. [Rudin, p. 98 # 3]. Let $\mathcal{M}$ be a metric space and $f : \mathcal{M} \to \mathbb{R}$ a continuous function. Denote by $Z(f)$ the *zero set of* f. These are the points $p \in \mathcal{M}$ where f is zero, $f(p) = 0$.

   a) Show that $Z(f)$ is a closed set.

   b) [See also Rudin, p. 101 #20] Given *any* set $E \in \mathcal{M}$, the distance of a point x to E is defined by
   $$h(x) = \rho_E(x) := \inf_{z \in E} d(x, z).$$
   Show that h is a uniformly continuous function.

   c) Use the previous part to show that given any *closed* set $E \in \mathcal{M}$, there is a continuous function that is zero on E and positive elsewhere.

8. [Rudin, p. 98 # 4]. Let f and g be continuous mappings of a metric space X into a metric space Y and let E be a dense subset of X.

   a) Prove that $f(E)$ is dense in $f(X)$.

   b) If $g(p) = f(p)$ for all $p \in E$, prove that $g(p) = f(p)$ for all points p in X. Thus, a continuous function is determined by its values in a dense subset of its domain.

9. [Rudin, p. 99 # 7]. For points $(x,y) \neq (0,0) \in \mathbb{R}^2$, define

$$f(x,y) = \frac{xy^2}{x^2 + y^4} \quad \text{and} \quad g(x,y) = \frac{xy^2}{x^2 + y^6},$$

   while define $f(0,0) = 0$ and $g(0,0) = 0$.

   a) Show that f is bounded in $\mathbb{R}^2$ but not continuous at the origin, while g is unbounded in every neighborhood of the origin and hence also not continuous there.

   b) Let $S \in \mathbb{R}^2$ be any straight line through the origin. Show that if the points (x,y) are stricted to lie on S, then both $f(x,y)$ and $g(x,y)$ are continuous. MORAL: It can be misleading to understand a function by only examining it on straight lines.

10. [Rudin, p. 99 # 8]. Let $E \subset \mathbb{R}$ be a set and $f : E \to \mathbb{R}$ be uniformly continuous.

    a) If E is a bounded set, show that $f(E)$ is a bounded set.

    b) If E is not bounded, give an example showing that $f(E)$ might not be bounded.

11. [Rudin, p. 99 # 13 or #11] *extension by continuity* Let X be a metric space, $E \subset X$ a dense subset, and $f : E \to \mathbb{R}$ a uniformly continuous function. Show that f has a unique continuous extension to all of X. That is, there is a unique continuous function $g : X \to \mathbb{R}$ with the property that $g(p) = f(p)$ for all $p \in X$. [REMARK: One generalize this by replacing $\mathbb{R}$ by any complete metric space.]

12. [Rudin, p. 101 # 23]. A real-valued function $f : (a,b) \to \mathbb{R}$ is called *convex* if

$$f(tx + (1-t)y) \leq t\,f(x) + (1-t)\,f(y) \quad \text{for all } x,y \in (a,b) \ \text{ and } \ 0 < t < 1.$$

    a) Prove that every convex function is continuous.

    b) Prove that every increasing convex function of a convex function is convex. Example: Assuming e^x is convex (it is), if f is convex then so is $e^{f(x)}$.

13. [Rudin, p. 101 # 24]. [CONTINUATION] Assume that $f : (a,b) \to \mathbb{R}$ is continuous and has the property that

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \quad \text{for all} \quad x,y \in (a,b).$$

Prove that f is convex. [REMARK: One can use this to give a short proof of the arithmetic-geometric mean inequality . Homework Set 3 #10].