Homework Set 4

DUE: Tues. Oct. 10, 2006. Late papers accepted until 1:00 Wednesday.

Math 508, Fall 2006 **Jerry L. Kazdan**

1. (Rudin, p.78 #6) Investigate the convergence or divergence of $\sum a_n$ if

$$\text{a). } a_n = \sqrt{n+1} - \sqrt{n} \qquad \text{b). } a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n} \qquad \text{c). } a_n = \frac{1}{1+z^n} \text{ (complex } z\text{)}$$

2. (Rudin p. 79 #8) Assume $a_n > 0$. If $\sum a_n$ converges and $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n b_n$ converges.

3. (Rudin p. 79 #9) Find the radius of convergence of each of the following power series.

$$\text{a). } \sum n^3 z^n, \qquad \text{b). } \sum \tfrac{2^n}{n!} z^n \qquad \text{c). } \sum n! z^n$$

4. Let $\{a_n\} \in \mathbb{R}$ be a bounded sequence. If $x > 1$ show that $\sum \dfrac{a_n}{n^x}$ converges absolutely.

5. (Rudin, p. 78 #7) If $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ converges, show that $\sum_{n=1}^{\infty} \dfrac{\sqrt{a_n}}{n}$ converges. [SUGGESTION: $0 \leq (x-y)^2 = x^2 - 2xy + y^2$ for all real x, y.]

6. Determine if the following series converges or diverges:

$$1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \dots$$

(the sign pattern is $++--++--++--++\dots$).

The next three problems are variations on just one idea.

7. Let $\{a_n\}$ be a sequence of real numbers with the property that

$$|a_{k+1} - a_k| \leq \tfrac{1}{2}|a_k - a_{k-1}|, \qquad k = 1, 2, \dots .$$

Show that this sequence converges to some real number.

8. a) Let $X_j, j = 1, 2, \ldots$ be a sequence of points in $\mathbb{R}^3$. If $\|X_{j+1} - X_j\| \le \dfrac{1}{j^4}$, show that these points converge.

   b) Let $\{X_j\}$ be a sequence of points in $\mathbb{R}^n$ with the property that

$$\sum_j \|X_{j+1} - X_j\| < \infty.$$

   Prove that the sequence $\{X_j\}$ converges. Give an example of a convergent sequence that does not have this property.

9. In a metric space M let $d(x,y)$ denote the distance. A sequence x_j is called a *fast Cauchy sequence* if $\sum_j d(x_{j+1}, x_j) < \infty$.

   a) In $\mathbb{R}$ give an example of a fast Cauchy sequence and also of a Cauchy sequence that is *not* fast.

   b) Show that every fast Cauchy sequence is indeed a Cauchy sequence.

   c) If there is a constant $0 < c < 1$ such that for all j

$$d(x_{j+1}, x_j) < c\, d(x_j, x_{j-1})$$

   show that x_j is a fast Cauchy sequence.

[Last revised: October 12, 2006]