Homework Set 3

DUE: Tues. Oct. 3, 2006. Late papers accepted until 1:00 Wednesday.

Math 508, Fall 2006 **Jerry L. Kazdan**

1. a) Calculate $\lim_{n\to\infty} \dfrac{5n+17}{n+2}$.

   b) Let $a_n := \dfrac{3n^2-2n+17}{n^2+21n+2}$. Calculate $\lim\limits_{n\to\infty} a_n$.

2. (Rudin, p.78 #2) Calculate $\lim\limits_{n\to\infty} \sqrt{n^2+n}-n$.

3. If $c>0$, show that $\dfrac{c^n}{n!} \to 0$ as $n\to\infty$.

4. If $b>1$ and $s\in\mathbb{R}$, show that $\dfrac{n^s}{b^n} \to 0$ as $n\to\infty$.

5. (Rudin, p.78 #3)Let $s_1=\sqrt{2}$, and $s_{n+1}=\sqrt{2+\sqrt{s_n}}$, $n=1,2,3,\ldots$. Prove that $\{s_n\}$ converges to some number s and that $s_n<2$.

6. (Rudin, p. 78 #5) Let $\{a_n\}$ and $\{b_n\}$ be any real sequences.

   a) Show that
   $$\limsup_{n\to\infty}(a_n+b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$
   provided the sum on the right is not of the form $\infty - \infty$.

   b) Give an explicit example where strict inequality can occur.

7. Let $p_k=(x_k,y_k)\in\mathbb{R}^2$, $k=1,2,\ldots$ be a sequence of points in the plane (with the usual Euclidean metric). Show that $\{p_k\}$ converges to $p=(x,y)$ if and only if $x_k\to x$ and $y_k\to y$.

8. [NEWTON] Let $A>0$ and $x_1>0$. Define $x_{n+1}=\dfrac{1}{2}\left(x_n+\dfrac{A}{x_n}\right)$. The following steps show that $x_n\to\sqrt{A}$.

a) Show that after the first term, the sequence $\{x_n\}$ is monotonically decreasing and that $x_2^2 \geq A$ (hence $x_n^2 \geq A$ for $n \geq 2$).

b) Show the x_n converge to some real number L and, using the definition of x_n, that $L^2 = A$.

9. Given a real sequence $\{a_k\}$, let $C_n = \dfrac{a_1 + \cdots + a_n}{n}$ be the sequence of averages (*arithmetic mean*).

   a) Give an example where the a_n's doesn't converge but the averages do converge.

   b) If the averages converge, must the a_n's be bounded? (Proof or counterexample)

   c) If a_k converges to A, show that also C_n converges, and to A.

   d) If $b_k \in \mathbb{R}$ are positive and $b_k \to B$, show that their *geometric mean* also converge to B, that is $[b_1 b_2 \cdots b_n]^{1/n} \to B$.

10. If $\{b_k\}$ is a sequence of positive numbers, prove the arithmetic-geometric mean inequality
$$[b_1 b_2 \cdots b_n]^{1/n} \leq \frac{b_1 + \cdots + b_n}{n}.$$
When does equality hold?

[Last revised: October 3, 2006]