Problem Set 2

DUE: Tues. Sept. 26, 2006. Late papers accepted until 1:00 Wednesday.

Math 508, Fall 2006 **Jerry L. Kazdan**

1. (Rudin, p.43 #2) A complex number *algebraic* if it is a root of a polynomial $a_0 z^n + \cdots + a_n$ whose coeffients are all integers. Prove that the set of all algebraic numbers is countable. [HINT: For every positive integer N there are only finitely many equations with $n + |a_0| + \cdots + |a_n| = N$.]

2. (Rudin, p.43 #5) Construct a bounded set of real numbers with exactly three limit points.

3. (Rudin, p.43 #6) Let E' be the set of limit points of a set E in a metric space. Show that E' is closed.

4. (Rudin, p.43 #10) Let X be any infinite set and for $p,\ q \in X$ define the function

$$d(p,q) = \begin{cases} 1 & \text{if } p \neq q, \\ 0 & \text{if } p = q \end{cases}$$

   Prove that this is a metric (although it is not very interesting). Which subsets are open? closed? compact?

5. (Rudin, p.45 #22) If x and y are real numbers, define
   $d_1(x,y) = (x-y)^2$; $\quad d_2(x,y) = \sqrt{|x-y|}$; $\quad d_3(x,y) = |x^2 - y^2|$;
   $d_4 = |x - 2y|$; $\quad d_5 = \frac{|x-y|}{1+|x-y|}$.
   Which of these define metrics? Justify your assertions.

6. (Rudin, p.44 #20) Are the closures and interiors of connected sets always connected? [Look at subsets of $\mathbb{R}^2$.]

7. (Rudin, p.45 #22) A metric space is called *separable* if it contains a countable dense subset. Show that $\mathbb{R}^2$ is separable. [HINT: Consider the set of points whose coordinates are rational.]

8. Define two real numbers x and y to be equal if $|x-y|$ is an integer, thus we have a topological circle whose circumference is one.

   Let α be an irrational real number, $0 < \alpha < 1$ and consider its integer multiples, α, 2α, 3α .... Show that this set is dense in $0 \leq x \leq 1$.

[Last revised: September 24, 2006]