Problem Set 1

DUE: Thurs. Sept. 14, 2006. Late papers accepted until 1:00 Friday.

Math508, Fall 2006 **Jerry L. Kazdan**

Many of these problems are from the Rudin text.

1. a) If $r(\neq 0)$ is a rational number and x is irrational, show that both $r+x$ and rx are *irrational*.

   b) Prove that there is no rational number whose square is 12.

2. (p. 22 #6) The point of this problem is, for any real $b>1$ and any real x to define b^x. So far we can only do this for integers x. First we extend this to rational x and then to all real x.

   Fix $b>1$. Let m,n,p,q be integers with $n>0$, $q>0$. Set $r=m/n=p/q$.

   a) Prove that $(b^m)^{1/n}=(b^p)^{1/q}$. Thus, it makes sense to define $b^r=(b^m)^{1/n}$.

   b) If r and s are rational, prove that $b^{r+s}=b^rb^s$.

   c) Ifd x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t\leq x$. Prove that for r rational
   $$b^r=\sup B(r).$$
   Hence it makes sense to *define* $b^x=\sup B(x)$ for all real x.

   d) With this definition, prove that for all real x,y: $b^{x+y}=b^xb^y$.

3. (p. 22 #7) If $b>1$ and $y>0$, prove there is a unique real x such that $b^x=y$ by completing the following outline. This x is called the *logarithm of y to the base b*.

   a) For any positive integer n, show that $b^n-1\geq n(b-1)$.

   b) Hence $b-1\geq n(b^{1/n}-1)$.

   c) If $t>1$ and $n>(b-1)/(t-1)$, show that $b^{1/n}<t$.

   d) If w is such that $b^w<y$, show that $b^{w+(1/n)}<y$ for sufficiently large n. [HINT: Apply the previous part with $t=y\cdot b^{-w}$].

   e) If $b^w>y$, show that $b^{w-(1/n)}>y$ for all sufficiently large integers n.

   f) Let A be the set of all w such that $b^w<y$. Show that the real number $x:=\sup A$ satisfies $b^x=y$.

   g) Prove that this x is unique.

4. Show that no order can be defined that makes the field of complex numbers into an ordered fielf. [HINT: -1 is the square of a complex number].

5. (p. 23 #12, #13) Let z, w, $z_1, \ldots, z_n$ be complex numbers

   a) Show that (*triangle inequality*)

$$|z_1 + \cdots + z_n| \leq |z_1| + \cdots + |z_n|.$$

   b) Show that $||z| - |w|| \leq |z - w|$.

6. (p. 23 #19) Suppose $a \in \mathbb{R}^k$, $b \in \mathbb{R}^k$, and $x \in \mathbb{R}^k$. Find all $c \in \mathbb{R}^k$ and $r > 0$ (depending on a and b) such that $|x - a| = 2|x - b|$ is satisfied if and only if $|x - c| = r$. [ANSWER: $3c = 4b - a$, $3r = 2|b - a|$].

[Last revised: September 12, 2006]