Math 508 **Exam 2** Jerry L. Kazdan
December 8, 2006 12:00 – 1:20

Directions This exam has two parts, Part A has 3 shorter problems (8 points each, so 24 points), Part B has 5 traditional problems (15 points each, so 75 points).
Closed book, no calculators – but you may use one $3'' \times 5''$ card with notes.

Part A: Short Problems (3 problems, 8 points each).

A–1. A continuous function $f : \mathbb{R} \to \mathbb{R}$ has the property that

$$\int_0^x f(t)\,dt = \cos(x)\,e^{-x} + C,$$

where C is some constant. Find both $f(x)$ and the constant C.

Solution: Letting $x = 0$, we find that $0 = 1 + C$, so $C = -1$. To compute f, use the fundamental theorem of calculus. Thus take the derivative of both sides

$$f(x) = \frac{d}{dx}[\cos(x)\,e^{-x} + C] = -\sin(x)\,e^{-x} - \cos(x)\,e^{-x}.$$

A–2. A function $h : \mathbb{R} \to \mathbb{R}$ with two continuous derivatives has the property that $h(0) = 2$, $h(1) = 0$, and h(3)=1. Prove there is at least one point c in the interval $0 < x < 3$ where $h''(c) > 0$ by finding some *explicit* $m > 0$ (such as $m = 3/2$) with $h''(c) \ge m$.

Solution: By the mean value theorem applied twice, there is some $a \in (0,\,1)$ and some $b \in (1,\,3)$ so that

$$h'(a) = \frac{h(1) - h(0)}{1 - 0} = -2, \qquad h'(b) = \frac{h(3) - h(1)}{3 - 1} = \frac{1}{2}.$$

Thus, by the mean value theorem again there is some $c \in (a,\,b)$ so that

$$h''(c) = \frac{h'(b) - h'(a)}{b - a} = \frac{\frac{1}{2} + 2}{b - a} > \frac{5/2}{3} = \frac{5}{6}.$$

A–3. Say a smooth function $u(x)$ satisfies $u'' - c(x)u = 0$ for $0 \le x \le 1$ (here $c(x)$ is some given contunuous function).

If $c(x) > 0$ everywhere, show that there is *no* point where $u(x)$ is both positive *and* has a local maximum.

If we also knew that $u(0) = 0$ and $u(1) = 0$, why can we conclude that $u(x) = 0$ for all $0 \le x \le 1$?

Solution: If u has a positive maximum at some point p, then $u''(p) \le 0$ and $u(p) > 0$. Consequently $u''(p) - c(p)u(p) < 0$, which contradicts $u'' - c(x)u = 0$.

If $u(0) = 0$ and $u(1) = 0$ but u is not identically zero, then u must either be positive or negative somewhere. Say u is positive somewhere (otherwise replace u by $-u$). Then since u is continuous on the compact set $[0,\,1]$, it has a positive maximum at some interior point. But we saw just above that this can't happen.

Part B: Traditional Problems (5 problems, 16 points each)

B–1. Given that two functions $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ are differentiable at a point $x=c$, prove that their product $h(x)=f(x)g(x)$ is also differentiable at $x=c$.

Solution:.

$$\begin{aligned}\frac{h(c+k)-h(c)}{k} =& \frac{[f(c+k)g(c+k)-f(c)g(c+k)]+[f(c)g(c+k)-f(c)g(c)]}{k}\\ =& \frac{f(c+k)-f(c)}{k}g(c+k)+f(c)\frac{g(c+k)-g(c)}{k}\end{aligned}$$

Now let $k\to 0$. Since f and g are both assumed to be differentiable at $x=c$, we see that h is differentiable there and get the usual formula: $h'(c)=f'(c)g(c)+f(c)g'(c)$.

B–2. Let $\alpha(t)$ and $\beta(s)$ describe smooth curves in $\mathbb{R}^3$ that do not intersect. Say the points $p=\alpha(t_0)$ and $q=\beta(s_0)$ minimize the distance between the curves. Show that the line from p to q is perpendicular to both of these curves.

Solution:. To avoid square roots, let Q be the square of the distance from $\alpha(t)$ to the point $\beta(s)$, so

$$Q(s,t)=\|\alpha(t)-\beta(s)\|^2=\langle\alpha(t)-\beta(s),\,\alpha(t)-\beta(s)\rangle$$

Then $Q(t,s)$ has its minimum at (t_0,s_0), Consequently both $\partial Q/\partial t=0$ and $\partial Q/\partial s=0$ at (t_0,s_0). But

$$\frac{\partial Q}{\partial t}=2\langle\alpha(t)-\beta(s),\,\alpha'(t)\rangle \quad\text{and}\quad \frac{\partial Q}{\partial s}=-2\langle\alpha(t)-\beta(s),\,\beta'(s)\rangle.$$

Evaluated at (t_0,s_0) the first gives $\alpha(t_0)-\beta(s_0)\perp\alpha'(t_0)$, whiile the second gives the other othogonality, $\alpha(t_0)-\beta(s_0)\perp\beta'(s_0)$.

B–3. Compute $\displaystyle\lim_{\lambda\to\infty}\int_0^1|\sin(\lambda x)|\,dx$.

Solution: First make the substitution $t=\lambda x$. and say $n\pi\le\lambda<(n+1)\pi$. Since $|\sin(\lambda t)|$ is periodic with period π, then the integral becomes

$$\begin{aligned}\frac{1}{\lambda}\int_0^\lambda|\sin t|\,dt =& \frac{1}{\lambda}\left[\int_0^\pi+\int_\pi^{2\pi}+\cdots+\int_{(n-1)\pi}^{n\pi}+\int_{n\pi}^\lambda|\sin t|\,dt\right]\\ =& \frac{1}{\lambda}\left[n\int_0^\pi\sin t\,dt+\int_{n\pi}^\lambda|\sin t|\,dt\right]=\frac{2n}{\lambda}+\frac{1}{\lambda}\int_{n\pi}^\lambda|\sin t|\,dt.\end{aligned}$$

Because $n\pi\le\lambda<(n+1)\pi$, then $\lambda/n\to\pi$ so $2n/\lambda\to 2/\pi$. Also

$$\frac{1}{\lambda}\int_{n\pi}^\lambda|\sin t|\,dt<\frac{1}{\lambda}\int_{n\pi}^{(n+1)\pi}\,dt=\frac{\pi}{\lambda}\to 0.$$

Consequently the limit is $2/\pi$.

B–4. Consider the linear space S of real sequences $x = (x_1, x_2, \ldots)$ with only a finite number of non-zero terms. Let $\|x\| := \max_j |x_j|$ (you may use without proof that this is actually a norm). Is this space complete with this norm? Justify your response.

Solution:. This space is *not* complete since a Cauchy sequence can tend to something with infinitely many non-zero terms. For instance, let

$$X_k = (1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots, \tfrac{1}{k}, 0, 0, \ldots),$$

which has k non-zero terms. If $n \geq k > N$, then

$$X_n - X_k = (0, \ldots, \tfrac{1}{k+1}, \tfrac{1}{k+2}, \tfrac{1}{n}, 0, 0, \ldots),$$

so

$$\|X_n - X_k\| = \tfrac{1}{k+1} < \tfrac{1}{N}.$$

Thus the X_k are a Cauchy sequence whose terms have an increasing number of non-zero elements, so it can't converge to an element in S.

B–5. For any two sets $S, T \subset \mathbb{R}^n$ with the usual Euclidean metric, define the *distance* between these sets as

$$\operatorname{dist}(S, T) = \inf_{x \in S,\, y \in T} \|x - y\|$$

a) Assume that S is compact, T is closed, and their intersection, $S \cap T$, is empty. Prove that there are points $p \in S$ and $q \in T$ with $\operatorname{dist}(S, T) = \|p - q\|$. In particular, $\operatorname{dist}(S, T) > 0$.

b) Does the above assertion remain true if S and T are any two disjoint closed subsets of $\mathbb{R}^n$? Proof or counterexample.

Solution: a). Let $m = \inf_{x \in S,\, y \in T} \|x - y\|$. Then there are $x_i \in S$ and $y_i \in T$ so that $\|x_i - y_i\| \to m$. We can assume that $\|x_i - y_i\| \leq m + 1$.

Since S is compact, the x_j have a convergent subsequence, x_{i_j} to some $p \in S$. To prove the corresponding assertion about the y_i, we first show they are bounded. Because S is compact, it lies in some ball, $\|x\| \leq R$. Therefore

$$\|y_i\| \leq \|y_i - x_i\| + \|x_i\| \leq m + 1 + R.$$

Consider the y_{i_j} corresponding to the x_{i_j}. Since it is bounded, it too has a convergent subsequence, $y_{i_{j_k}}$. Because T is closed, this subsequence converges to some point $q \in T$. Using $\|x_{i_{j_k}} - y_{i_{j_k}}\| \to m$, we see that

$$\|p - q\| = \lim \|x_{i_{j_k}} - y_{i_{j_k}}\| = m.$$

Because S and T are disjoint, $p \neq q$ so $m = \|p - q\| > 0$.

b). The assertion is *false* if we only assume the sets S and T are closed. One example is $S = \{(x, y) \in \mathbb{R}^2 \mid y \geq \frac{1}{1+x^2}\}$ and $T = \{(x, y) \in \mathbb{R}^2 \mid y \leq 0\}$. Then $\operatorname{dist}(S, T) = 0$. It is easy to cook up many examples.

In a general metric space, the assertion in part a) is false, even if you also assume T bounded. For example, in ℓ_2, let $S = \{0\}$ (the origin) and $T = \{(1 + \frac{1}{n})e_n,\ n = 1, 2, 3, \ldots\}$ where e_1, e_2, e_3, … are the standard basis vectors. Then $\operatorname{dist}(S, T) = 1$ although there are no points $p \in S$, $q \in T$ with $\|p - q\| = 1$. The problem is that although T is closed and bounded, it has no convergent subsequence.

In a general metric space, the assertion in part a) is true if *both* S and T are compact since a continuous function on a compact set achieves its minimum. One can use this to give a slightly different proof of part a) as follows. As observes above, if $m = \inf_{x \in S,\, y \in T} \|x - y\|$, then to find the minimum distance, we need only use the points $y \in T$ that are within distance $m + 1$ from S, that is,

$$Q := \{y \in T \mid | \, d(x, y) \leq m + 1 \text{ for all } x \in S\}$$

But since S is compact, it is bounded, so Q is bounded (and closed). Since $Q \in \mathbb{R}^n$, it is compact. Thus for $x \in S$ and $y \in Q$, the function $d(x, y)$ is a continuous function on a compace set so it achieves its minimum at some point of the set. Because the sets are disjoint, this minimum is strictly positive.