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Math 508
December 8, 2006

Exam 2

Jerry L. Kazdan
12:00 – 1:20

DIRECTIONS This exam has two parts, Part A has 3 shorter problems (8 points each, so 24 points), Part B has 5 traditional problems (15 points each, so 75 points).
Closed book, no calculators – but you may use one $3'' \times 5''$ card with notes.

Part A: Short Problems (3 problems, 8 points each).

A–1. A continuous function $f : \mathbb{R} \to \mathbb{R}$ has the property that

$$\int_0^x f(t)\,dt = \cos(x)\,e^{-x} + C,$$

where C is some constant. Find both $f(x)$ and the constant C.

A–2. A function $h : \mathbb{R} \to \mathbb{R}$ with two continuous derivatives has the property that $h(0) = 2$, $h(1) = 0$, and h(3)=1. Prove there is at least one point c in the interval $0 < x < 3$ where $h''(c) > 0$ by finding some *explicit* $m > 0$ (such as $m = 2/3$) with $h''(c) \ge m$.

A–3. Say a smooth function $u(x)$ satisfies $u'' - c(x)u = 0$ for $0 \le x \le 1$ (here $c(x)$ is some given contunuous function).

If $c(x) > 0$ everywhere, show that there is *no* point where $u(x)$ is both positive *and* has a local maximum.

If we also knew that $u(0) = 0$ and $u(1) = 0$, why can we conclude that $u(x) = 0$ for all $0 \le x \le 1$?

Part B: Traditional Problems (5 problems, 16 points each)

B–1. Given that two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are differentiable at a point $x = c$, prove that their product $h(x) = f(x)g(x)$ is also differentiable at $x = c$.

B–2. Let $\alpha(t)$ and $\beta(s)$ describe smooth curves in $\mathbb{R}^3$ that do not intersect. Say the points $p = \alpha(t_0)$ and $q = \beta(s_0)$ minimize the distance between the curves. Show that the line from p to q is perpendicular to both of these curves.

B–3. Compute $\displaystyle\lim_{\lambda\to\infty} \int_0^1 |\sin(\lambda x)|\,dx$.

B–4. Consider the linear space S of real sequences $x = (x_1, x_2, \ldots)$ with only a finite number of non-zero terms. Let $\|x\| := \max_j |x_j|$ (you may use without proof that this is actually a norm). Is this space complete with this norm? Justify your response.

B–5. For any two sets $S, T \subset \mathbb{R}^n$ with the usual Euclidean metric, define the *distance* between these sets as

$$\operatorname{dist}(S,T) = \inf_{x\in S,\, y\in T} \|x-y\|$$

a) Assume that S is compact, T is closed, and their intersection, $S \cap T$, is empty. Prove that there are points $p \in S$ and $q \in T$ with $\operatorname{dist}(S,T) = \|p-q\|$. In particular, $\operatorname{dist}(S,T) > 0$.

b) Does the above assertion remain true if S and T are any two disjoint closed subsets of $\mathbb{R}^n$? Proof or counterexample.