Math 508 — **Exam 1** — Jerry L. Kazdan

October 12, 2006 — 12:00 – 1:20

DIRECTIONS This exam has three parts, Part A has 4 problems asking for Examples (20 points, 5 points each), Part B asks you to describe some sets (20 points), Part C has 4 traditional problems (60 points, 15 points each).
Closed book, no calculators – but you may use one $3'' \times 5''$ card with notes.

Part A: Examples (4 problems, 5 points each). Give an example of an infinite set in a metric space (perhaps $\mathbb{R}$) with the specified property.

A–1. Bounded with exactly two limit points.

Solution: The set $\{(-1)^n(1+\frac{1}{n}),\ n = 1, 2, 3, \ldots\}$ in $\mathbb{R}$.

A–2. Containing all of its limit points.

Solution: Lots of exmples: 1). The empty set. 2). All of $\mathbb{R}$. 3). The point $\{0\} \in \mathbb{R}$. 4). The closed interval $\{0 \le x \le 1 \text{ in } \mathbb{R}\}$.

A–3. Distinct points $\{x_j,\ j = 1, 2, 3, \ldots\}$ with $x_i \ne x_j$ for $i \ne j$ that is compact.

Solution: The following subset of the real numbers: $\{0\} \cup \{\frac{1}{n},\ n = 1, 2, 3, \ldots\}$.

A–4. Closed and bounded but not compact.

Solution: The closed unit ball $\|x\| \le 1$ in ℓ_2. The standard basis vectors $e_1 = (1, 0, 0, \ldots)$, $e_2 = (0, 1, 0, 0, \ldots)$, etc have no convergent subsequence.

Another example: the real numbers $\{x \in \mathbb{R} \mid 0 \le x \le 1\}$ with the discrete metric: $d(x, y) = 1$ for $x \ne y$, $d(x, x) = 0$.

Part B: Classify sets (20 points) For each of the following sets, **circle** the listed properties it has:

a) $\{1 + \frac{1}{n} \in \mathbb{R},\ n = 1, 2, 3, \ldots\}$ open closed [bounded] compact [countable]

b) $\{1\} \cup \{1 + \frac{1}{n} \in \mathbb{R},\ n = 1, 2, 3, \ldots\}$

open [closed] [bounded] [compact] [countable]

c) $\{(x, y) \in \mathbb{R}^2 : 0 < y \le 1\}$ open closed bounded compact countable

d) $\{(x, y) \in \mathbb{R}^2 : x = 0\}$ open [closed] bounded compact countable

e) $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ open $\boxed{\text{closed}}$ $\boxed{\text{bounded}}$ $\boxed{\text{compact}}$ countable

f) $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$ open $\boxed{\text{closed}}$ $\boxed{\text{bounded}}$ $\boxed{\text{compact}}$ countable

g) $\{(x,y) \in \mathbb{R}^2 : y > x^2\}$ $\boxed{\text{open}}$ closed bounded compact countable

h) $\{(k,n) \in \mathbb{R}^2 : k, n$ any positive integers$\}$

open $\boxed{\text{closed}}$ bounded compact $\boxed{\text{countable}}$

Part C: Traditional Problems (4 problems, 20 points each)

C–1. In $\mathbb{R}$, if $a_n \to A$ and $b_n \to B$, show that the product $a_n b_n \to AB$.

Solution: Let $p_n = a_n - A \to 0$, $q_n = b_n - B \to 0$. Then

$$a_n b_n = (p_n + A)(q_n + B) = p_n q_n + A q_n + B p_n + AB.$$

Using that for convergent sequences x_n and y_n we know $\lim(x_n + y_n) = \lim x_n + \lim y_n$ and $\lim(c x_n) = c \lim x_n$, we see that it is enough to show that $p_n q_n \to 0$. Given $\epsilon > 0$ (which we may assume satisfies $\epsilon < 1$), pick N so that if $n > N$ then $|p_n| < \epsilon$ and $|q_n| < \epsilon$. Consequently $|p_n q_n| < \epsilon^2 < \epsilon$.

C–2. Given a real sequence $\{a_k\}$, let $C_n = \dfrac{a_1 + \cdots + a_n}{n}$ be the sequence of averages (*arithmetic mean*). If a_k converges to A, show that the averages C_n also converge to A.

Solution: Letting $B_n = a_n - A \to 0$, I could reduce to the case $A = 0$. Instead, for variety I proceed directly. Note that

$$C_n - A = \frac{a_1 + \cdots + a_n}{n} - A = \frac{(a_1 - A) + \cdots + (a_n - A)}{n}$$

Given any $\epsilon > 0$, pick N so that if $n > N$ then $|a_n - A| < \epsilon$. Then write

$$C_n - A = \underbrace{\frac{(a_1 - A) + \cdots + (a_N - A)}{n}}_{I_n} + \underbrace{\frac{(a_{N+1} - A) + \cdots + (a_n - A)}{n}}_{J_n}.$$

Now

$$|J_n| < \frac{[n - (N+1)]\epsilon}{n} \le \frac{n\epsilon}{n} = \epsilon \quad \text{for any} \quad n > N.$$

We will show that by choosing n even larger, we can make $|I_n| < \epsilon$. Since the sequence $a_n - A$ converges, it is bounded, so for some M we have $|a_n - A| < M$. Thus for n sufficientnly large

$$|I_n| < \frac{NM}{n} < \epsilon.$$

Consequently, $|C_n - A| \le |I_n| + |J_n| < 2\epsilon$.

C–3. Let K_j, $j = 1, 2, \ldots$ be compact sets in a metric space. Give a proof or counterexample for each of the following assertions.

a) $K_1 \cap K_2$ is compact.

Solution: True. Since compact sets are closed, then $K_1 \cap K_2$ is a closed subset of the compact set K_1, and hence compact.

b) $K_1 \cup K_2$ is compact.

Solution: True. Let $\{U_\alpha\}$ be any open cover of $K_1 \cup K_2$. A finite number of these, say $\{V_1, \ldots, V_k\}$, cover K_1, and $\{W_1, \ldots, W_n\}$, cover K_2. Then $\{V_1 \cup \ldots \cup V_k \cup W_1 \cup \ldots \cup W_n\}$ is the desired finite cover of $K_1 \cup K_2$.

c) $\bigcup_{j=1}^{\infty} K_j$ is compact.

Solution: Counterexample. The non-negative real numbers $\{x \geq 0\}$ is the union of the compact sets (closed intervals) $K_j = \{j - 1 \leq x \leq j;\ j = 1, 2, \ldots\}$. Since this set is not bounded, it is not compact.

C–4. In a *complete* metric space M, let $d(x, y)$ denote the distance. Assume there is a constant $0 < c < 1$ so that the sequence x_k satisfies

$$d(x_{n+1},\, x_n) < cd(x_n,\, x_{n-1}) \quad \text{for all} \quad n = 1, 2, \ldots.$$

a) Show that $d(x_{n+1}, x_n) < c^n d(x_1, x_0)$.

Solution: Since $d(x_2,\, x_1) < cd(x_1,\, x_0)$, then

$$d(x_3,\, x_2) < cd(x_2,\, x_1) < c^2 d(x_1,\, x_0).$$

Using this,

$$d(x_4,\, x_3) < cd(x_3,\, x_2) < c^3 d(x_1,\, x_0).$$

The induction to the general case is obvious.

b) Show that the $\{x_k\}$ is a Cauchy sequence.

Solution: Say $n > k$. Then using the previous part and that $0 < c < 1$

$$\begin{aligned} d(x_n,\, x_k) &\leq d(x_n,\, x_{n-1}) + \ldots + d(x_{k+1},\, x_k) \\ &\leq (c^{n-1} + c^{n-2} + \cdots + c^k)\, d(x_1,\, x_0) \\ &\leq (c^k(1 + c + c^2 + c^3 + \ldots)\, d(x_1,\, x_0) = \frac{c^k}{1-c}\, d(x_1,\, x_0). \end{aligned}$$

Pick N so that $c^N < \epsilon$. If $n > k > N$ then

$$d(x_n,\, x_k) \leq < \frac{\epsilon}{1-c}\, d(x_1,\, x_0).$$

c) Show that there is some $p \in M$ so that $\lim_{n\to\infty} x_k = p$.

Solution: Since the metric space is complete, there is a point p in the metric space to which the Cauchy sequence x_k converges.