Math 508 — October 12, 2006

Exam 1

Jerry L. Kazdan — 12:00 – 1:20

DIRECTIONS This exam has three parts, Part A has 4 problems asking for Examples (20 points, 5 points each), Part B asks you to describe some sets (20 points), Part C has 4 traditional problems (60 points, 15 points each).
Closed book, no calculators – but you may use one $3'' \times 5''$ card with notes.

Part A: Examples (4 problems, 5 points each). Give an example of an infinite set in a metric space (perhaps $\mathbb{R}$) with the specified property.

A–1. Bounded with exactly two limit points.

A–2. Containing all of its limit points.

A–3. Distinct points $\{x_j\}$, $j = 1, 2, \ldots$ with $x_i \neq x_j$ for $i \neq j$ that is compact.

A–4. Closed and bounded but not compact.

Part B: Classify sets (20 points) For each of the following sets, **circle** the listed properties it has:

a) $\{1 + \frac{1}{n} \in \mathbb{R},\ n = 1, 2, 3, \ldots\}$ open closed bounded compact countable

b) $\{1\} \cup \{1 + \frac{1}{n} \in \mathbb{R},\ n = 1, 2, 3, \ldots\}$
open closed bounded compact countable

c) $\{(x, y) \in \mathbb{R}^2 : 0 < y \leq 1\}$ open closed bounded compact countable

d) $\{(x, y) \in \mathbb{R}^2 : x = 0\}$ open closed bounded compact countable

e) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ open closed bounded compact countable

f) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ open closed bounded compact countable

g) $\{(x, y) \in \mathbb{R}^2 : y > x^2\}$ open closed bounded compact countable

h) $\{(k, n) \in \mathbb{R}^2 : k, n \text{ any positive integers}\}$
open closed bounded compact countable

Part C: Traditional Problems (4 problems, 20 points each)

C–1. In $\mathbb{R}$, if $a_n \to A$ and $b_n \to B$, show that the product $a_n b_n \to AB$.

C–2. Given a real sequence $\{a_k\}$, let $C_n = \dfrac{a_1 + \cdots + a_n}{n}$ be the sequence of averages (*arithmetic mean*). If a_k converges to A, show that the averages C_n also converge to A.

C–3. Let K_j, $j = 1, 2, \ldots$ be compact sets in a metric space. Give a proof or counterexample for each of the following assertions.

a) $K_1 \cap K_2$ is compact.

b) $K_1 \cup K_2$ is compact.

c) $\bigcup_{j=1}^{\infty} K_j$ is compact.

C–4. In a *complete* metric space M, let $d(x, y)$ denote the distance. Assume there is a constant $0 < c < 1$ so that the sequence x_k satisfies

$$d(x_{n+1}, x_n) < c d(x_n, x_{n-1}) \quad \text{for all} \quad n = 1, 2, \ldots.$$

a) Show that $d(x_{n+1}, x_n) < c^n d(x_1, x_0)$.

b) Show that the $\{x_k\}$ is a Cauchy sequence.

c) Show that there is some $p \in M$ so that $\lim_{n\to\infty} x_k = p$.