Miscelaneous Problems

Math 504 – 505 Jerry L. Kazdan

In the following, when we say a function is smooth, *we mean that all of its derivatives exist and are continuous.*

1. Prove that
$$(1+x)^n > 1+nx$$
for every integer $n \geq 2$ and all $x > -1$, $x \neq 0$. (*Hint*: Induction on n.)

2. a) Prove that the positive integers and the positive even integers have the same cardinality.

   b) Prove that the intervals $\{0 \leq x \leq 1\}$ and $\{0 \leq x \leq 2\}$ have the same cardinality.

   c) Prove that the intervals $\{0 \leq x < 1\}$ and $\{0 \leq x \leq 1\}$ have the same cardinality.

   d) Prove that the intervals $\{0 < x < 1\}$ and the set of all real numbers have the same cardinality.

   e) Prove that the points in the interval $\{x \in \mathbb{R} \mid \ 0 \leq x \leq 1\}$ and the square $\{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$ have the same cardinality.

   f) Prove that the cardinality of the integers and real numbers are different.

3. If $c_1 + \cdots + c_n = 1$ and $c_j > 0$, let $\overline{w} = c_1w_1 + c_2w_2 + \cdots + c_nw_n$ be the *weighted average* of real numbers $w_1, \ldots, w_n$. Clearly this average lies between the max and min of the w_j. If $0 < \gamma \leq c_j$ for all j, prove the following more precise quantitative version of this:
$$\gamma w_{\max} + (1-\gamma)w_{\min} \leq \overline{w} \leq (1-\gamma)w_{\max} + \gamma w_{\min}.$$

4. Every day you wait on a certain corner for a bus. There are two different bus lines that stop at this corner, each going by your destination. The bus lines both run every 10 minutes, but they begin at varying times each day.

   What is the *average* time you must wait until a bus comes?

5. Given a hexagon whose sides have length r_1, r_2, r_3, r_4, r_5, and r_6.

   a) If each of the interior angles of the hexagon are 120 degrees, show that $r_1 + r_2 = r_4 + r_5$, $r_2 + r_3 = r_5 + r_6$, and $r_3 + r_4 = r_6 + r_1$.

b) Prove the converse: given any positive real numbers $r_1, \dots r_6$ that satisfy the above relationships, there is a hexagon whose sides have these lengths and whose interior angles are 120 degrees.

6. Let Γ_0 be a closed curve in the plane that encloses a convex region. Let Γ_r be the "parallel" curve one obtains by moving out a distance r along the outer normal. Discover and then prove a formula relating the length of Γ_r and Γ_0. [SUGGESTION: Try the special cases where Γ_0 is a circle, rectangle, and a convex polygon].

7. Given $V := (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$, let the matrix $A := \left(v_i v_j\right)$, so $a_{ij} = v_i v_j$.

   a) In the special case where v is a unit vector, show that for any vector x, the vector Ax is the orthogonal projection of x in the direction of v.

   b) Compute $\ker A$, $\operatorname{im} A$ (the *image* of A), as well as the eigenvalues and corresponding eigenvectors of A.

   c) Repeat the previous part for $B := I + A$ and also find B^{-1} (why does it exist?).

   d) If $W := (w_1, w_2, \dots, w_n) \in \mathbb{R}^n$, let $C := \left(v_i w_j\right)$. Using part c) as a model, what can you say about $D := I + C$?

   e) If $u(x_1, x_2, \dots, x_n)$ is a given smooth function, let $u'' := \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)$ be its second derivative (Hessian) matrix. Find all solutions of $\det(u'') = 1$ in the special case where $u = u(r)$ depends only on $r = \sqrt{x_1^2 + \cdots x_n^2}$, the distance to the origin.

8. Let $x = (x_1, x_2, \dots, x_n)$ and A be a square matrix with $\det A = 1$. If $u(x)$ satisfies $\det(u'') = 1$ (see above), and $v(x) := u(Ax)$, show that $\det(v'') = 1$ also. [Remark: the differential operator $\det(u'')$ is interesting because its symmetry group is so large.]

9. a) Make sense of the following: "Let $\mathcal{D}_t \subset \mathbb{R}^2$ be a family of bounded regions in the plane with smooth boundaries. These regions depend smoothly on a real parameter t."

   b) As a test of the effectiveness of your definition, use it to compute the derivative of

$$J(t) := \iint_{\mathcal{D}_t} f(x,y)\, dx\, dy,$$

   at $t = 0$ where $f(x,y)$ is a given smooth function and $\mathcal{D}_t$ is the family of ellipsoids $x^2 + (1+3t)y^2 = 1$.

10. Given any closed set Q in the plane, show there is a continuous function $f \geq 0$ with the property that $f = 0$ only on Q. [In fact, there is a smooth function with this same property, although it is more difficult to construct].

11. Let S and T be sets and $f : S \to T$.

    a) Show that f is injective if and only if there is a *left inverse* $\phi : T \to S$ such that $\phi \circ f = \mathrm{id}_S$, where id_S is the identity map on S. If S and T are linear spaces and f is a linear map, show that the left inverse ϕ is also a linear map.

    b) Show that f is surjective if and only if there is a *right inverse* $\psi : T \to S$ such that $f \circ \phi = \mathrm{id}_T$.

    c) If f is bijective, is it true that $\phi = \psi$? Proof or counterexample.

    d) If V and W are linear spaces and $L : V \to W$ is a bijective linear map, show that L has a (two-sided) inverse M that is also a linear map.

12. a) Let A be a symmetric matrix with eigenvalue λ and corresponding eigenvector v, $Av = \lambda v$. Show that
    $$\lambda = \frac{\langle v, Av\rangle}{\|v\|^2}.$$

    b) If A is positive definite with positive definite square root P, so $A = P^2$, show that
    $$\lambda = \frac{\|Pv\|^2}{\|v\|^2}.$$

    c) A standard ingredient in many problems involves the eigenvalues λ and corresponding eigenfunctions u of the Laplacian $\Delta = \nabla^2$,
    $$-\Delta u = \lambda u \quad \text{in} \quad \mathcal{D} \qquad \text{with} \qquad u = 0 \quad \text{on} \quad \mathcal{B},$$
    Here $\mathcal{D}$ in $\mathbb{R}^2$ is a bounded region with boundary $\mathcal{B}$. As usual, to be useful one wants numbers λ so that there is a solution u other than the trivial solution $u \equiv 0$. Show that
    $$\lambda = \frac{\iint_{\mathcal{D}} |\nabla u|^2 \, dA}{\iint_{\mathcal{D}} u^2 \, dA}.$$
    [SUGGESTION: Use the divergence theorem].
    In particular, deduce that $\lambda > 0$.

13. Suppose ξ is an irrational number. Prove that for all $\varepsilon > 0$, there are integers a and b such that $0 < |a - b\xi| < \varepsilon$. Is this true if ξ is rational?

14. For a real number $c \geq 0$, let $[c]$ be its fractional part. Given $\alpha \in \mathbb{R}$, let $J(\alpha) = \{[n\alpha], n = 1, 2, 3, \ldots\}$. Show that $J(\alpha)$ is dense in the interval $[0,1]$ if and only if α is irrational.

15. a) Compute

$$\iint_{\mathbb{R}^2} \frac{dx\,dy}{(1+4x^2+9y^2)^2}, \quad \iint_{\mathbb{R}^2} \frac{dx\,dy}{(1+x^2+2xy+5y^2)^2}, \quad \iint_{\mathbb{R}^2} \frac{dx\,dy}{(1+5x^2-4xy+5y^2)^2}.$$

b) Compute $\displaystyle\iint_{\mathbb{R}^2} \frac{dx_1\,dx_2}{[1+\langle x, Cx\rangle]^2}$, where C is a positive definite (symmetric) 2×2 matrix, and $x = (x_1, x_2) \in \mathbb{R}^2$.

c) Let $h(t)$ be a given function and say you know that $\int_0^\infty h(t)\,dt = \alpha$. If C be a positive definite 2×2 matrix. Show that

$$\iint_{\mathbb{R}^2} h(\langle x, Cx\rangle)\, dA = \frac{\pi\alpha}{\sqrt{\det C}}.$$

d) Compute $\displaystyle\iint_{\mathbb{R}^2} e^{-(5x^2-4xy+5y^2)}\,dx\,dy.$

e) Compute $\displaystyle\iint_{\mathbb{R}^2} e^{-(5x^2-4xy+5y^2-2x+3)}\,dx\,dy.$

f) Generalize part c) to obtain a formula for

$$\iint_{\mathbb{R}^n} h(\langle x, Cx\rangle)\, dV = \frac{\pi\alpha}{\sqrt{\det C}},$$

where now C be a positive definite $n\times n$ matrix. The answer will involve some integral involving h and also the "area" of the unit sphere $S^{n-1} \hookrightarrow \mathbb{R}^n$.

16. Prove Descartes' "Rule of Signs" concerning the number of positive real roots of a real polynomial.

17. A straight line ℓ is tangent to the cubic $y = x^3 + bx^2 + cx + d$. Let Q be the bounded region between the line and the cubic. Let ℓ' be the (unique) line that is parallel to ℓ and is also tangent to thee same cubic. This line also defines a region, R, between itseld and the cubic.

a) Show that Q and R have the same areas.

b) What else can you say about the relationship between Q and R?

18. Let $\{L_1, L_2\}$ and $\{M_1, M_2\}$ be two pairs of straight lines in $\mathbb{R}^2$. Find a necessary and sufficient condition that bthere is an isometry of $\mathbb{R}^2$ that maps $\{L_1, L_2\}$ to $\{M_1, M_2\}$.

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