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Math 504 **Exam 1** Jerry L. Kazdan

October 22, 2009 10:30 — 11:50

DIRECTIONS: Part A has 5 shorter problems (5 points each) while Part B has 6 traditional problems (10 points each). To receive full credit your solution should be clear and correct. You have 1 hour 20 minutes. Closed book, no calculators, but you may use one 3 ×5 with notes on both sides. Please box your answers.

PART A: SHORTER PROBLEMS 25 POINTS (5 POINTS EACH)

Score	
A-1	
A-2	
A-3	
A-4	
A-5	
B-1	
B-2	
B-3	
B-4	
B-5	
B-6	
Total	

A-1. Let c be any complex number. Show that $\lim_{n\to\infty} \frac{c^n}{n!} = 0$.

A-2. Show that $\sin x$ is not a polynomial.

A-3. Let A be a matrix, not necessarily square. Say $\mathbf{V}$ and $\mathbf{W}$ are particular solutions of the equations $A\mathbf{V} = \mathbf{Y}_1$ and $A\mathbf{W} = \mathbf{Y}_2$, respectively, while $\mathbf{Z} \neq 0$ is a solution of the homogeneous equation $A\mathbf{Z} = 0$. Answer the following in terms of $\mathbf{V}$, $\mathbf{W}$, and $\mathbf{Z}$.

a) Find some solution of $A\mathbf{X} = 3\mathbf{Y}_1 - 5\mathbf{Y}_2$.

b) Find *another* solution of $A\mathbf{X} = 3\mathbf{Y}_1 - 5\mathbf{Y}_2$.

c) If A is a square matrix, then $\det A = ?$

A-4. Let $A : \mathbb{R}^3 \to \mathbb{R}^2$ and $B : \mathbb{R}^2 \to \mathbb{R}^3$, so $BA : \mathbb{R}^3 \to \mathbb{R}^3$. Show that BA can *not* be invertible.

A-5. Let a smooth function $g(x)$ have the three properties: $g(0) = 2 \quad g(1) = 0 \quad g(4) = 6$. Show that at some point $0 < c < 4$ one has $g''(c) > 0$. To be more specific, find a number $m > 0$ so that $g''(c) \ge m > 0$.

PART B: STANDARD PROBLEMS 60 POINTS (10 POINTS EACH)

B-1. Say you have k linear algebraic equations in n variables; in matrix form we write $AX = Y$. Give a proof or counterexample for each of the following assertions.

a) If $n = k$ there is always *at most one* solution.

b) If $n > k$ you can *always* solve $AX = Y$.

c) If $n > k$ the nullspace of A has dimension greater than zero.

d) If $n < k$ then for *some* Y there is *no* solution of $AX = Y$.

e) If $n < k$ the *only* solution of $AX = 0$ is $X = 0$.

B-2. Let c_n be a sequence of real numbers that converges to C. Show that their "average" $S_n := \dfrac{c_1 + c_2 + \cdots + c_n}{n}$ also converges to C.

B-3. Compute $\displaystyle\iint_{\mathbb{R}^2} \frac{1}{[1 + (2x - y + 1)^2 + (x + y + 3)^2]^2}\, dx\, dy$.

B-4. Is $k(x) = \sqrt{x}$ uniformly continuous in the set $\{x \geq 0\}$? Justify your assertions.

B-5. If the sequence $\{a_n\}$ is bounded and $c > 1$, show that the series $\sum_{n=1}^{\infty} \frac{a_n}{n^x}$ converges absolutely and uniformly in the interval $c \leq x < \infty$.

B-6. Let $u(x,y,t)$ be a solution of the heat equation $u_t = \Delta u$ for (x,y) in a smoothly bounded open set $\mathcal{D} \subset \mathbb{R}^2$ and $t \geq 0$. Assume that the temperature $u(x,y,t) = 0$ for all points (x,y) on the boundary $\mathcal{B}$ of $\mathcal{D}$ for all $t \geq 0$.

a) Let $E(t) := \frac{1}{2} \iint_{\mathcal{D}} u^2(x,y,t)\, dx\, dy$. Show that $dE/dt \leq 0$.

b) Use this to show that with these zero boundary conditions, if the initial temperature is zero, $u(x,y,0) = 0$ for all $(x,y) \in \mathcal{D}$, then $u(x,y,t) = 0$ for *all* $t \geq 0$.