Math 504 | Exam Solutions | Jerry L. Kazdan

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Exam Solutions

DIRECTIONS This exam has two parts, the first has four short computations (*5 points each*) while the second has seven traditional problems (*10 points each*).

Part A: Short Computations (4 problems, 5 points each)

A–1. Find a real 2×2 matrix A (other than $A=\pm I$) such that $A^4=I$.

Solution Two solutions are $A=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ (reflection across the x-axis and $A=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ (reflection across the line $y=x$). Both of these satisfy $A^2=I$ so clearly $A^4=I$.

A more interesting example that does *not* satisfy $A^2=I$ is $A=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ (rotation by 90 degrees).

A–2. Find a function $u(x,y)$ satisfying $\dfrac{\partial u}{\partial x}-2u=0$ with $u(0,y)=\sin(3y)$.

Solution This is just the ODE $u'-2u=0$ with y as a parameter. The general solution (say using separation of variables) is $u(x,y)=C(y)e^{2x}$. But $\sin(3y)=u(0,y)=C(y)$. Thus $\boxed{u(x,y)=\sin(3y)e^{2x}}$.

A–3. Say $T(x,y,z)=x^2+xy+y^3-z^2$ gives the temperature at the point (x,y,z). At the point $(1,1,1)$, in which direction should one move so that the temperature increases fastest?

Solution The gradient of a function is a vector pointing in the direction where the function increases most rapidly. Since

$$\nabla f(x,y,z)=(2x+y,\ x+3y^2,\ -2z),$$

the desired direction at $(1,1,1)$ is $(3,\ 4,\ -2)$. If you prefers, you can use a unit vector in this direction.

A–4. Compute $J:=\displaystyle\iint_{\mathbb{R}^2}\frac{1}{[1+(2x+y+1)^2+(x-y+3)^2]^2}\,dx\,dy$.

Solution Make the change of variables

$$u=2x+y+1,\qquad v=x-y+3.$$

Then

$$du\,dv=\left|\det\begin{pmatrix}2&1\\1&-1\end{pmatrix}\right|dx\,dy=3\,dx\,dy$$

so $dx\,dy=\frac{1}{3}\,du\,dv$. Thus

$$J=\tfrac{1}{3}\iint_{\mathbb{R}^2}\frac{1}{[1+u^2+v^2]^2}\,du\,dv,$$

which is computed using polar coordinates in the uv plane

$$J=\tfrac{1}{3}\int_0^{2\pi}\left[\int_0^\infty\frac{1}{[1+r^2]^2}\,r\,dr\right]d\theta=\frac{\pi}{3}.$$

Part B: Traditional Problems (7 problems, 10 points each)

B–1. If $A = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$, find an invertible matrix C such that $D := C^{-1}AC$ is a diagonal matrix. Compute A^{50}.

Solution This is routine. D has the eigenvalues of A and the columns of C are the corresponding eigenvectors of A. Since A is a symmetric matrix, by using unit eigenvectors we even have that C is an orthogonal matrix (so it's inverse is easier to compute). The upshot is

$$\lambda_1 = 5, \quad \lambda_2 = -3, \qquad v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

so

$$D = \begin{pmatrix} 5 & 0 \\ 0 & -3 \end{pmatrix} \qquad C = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad .$$

To compute A^{50}, use $A = CDC^{-1}$ to find

$$A^{50} = CD^{50}C^{-1} = \frac{1}{2}\begin{pmatrix} 5^{50} + (-3)^{50} & 5^{50} - (-3)^{50} \\ 5^{50} - (-3)^{50} & 5^{50} + (-3)^{50} \end{pmatrix}.$$

B–2. Let $T : \mathbb{R}^n \to \mathbb{R}^k$ be a real matrix (not necessarily square). If the nullspace of T is $\{0\}$, show that the matrix T^*T is invertible and positive definite.

Solution First we show that T^*T is positive definite. For any vector $x \in \mathbb{R}^n$

$$\langle x, T^*Tx \rangle = \langle Tx, TX \rangle = \|Tx\|^2 \geq 0.$$

This also shows that $\langle x, T^*Tx \rangle = 0$ only when $Tx = 0$. Since the nullspace of T is $\{0\}$, this occurs only when $x = 0$, so T^*T is positive definite.

The above computation contains the proof that the nullspace of the square matrix T^*T is $\{0\}$; thus it is invertible.

B–3. Let a_n be a bounded sequence of real numbers. If $c > 1$, show that the series $\sum \frac{a_n}{n^x}$ converges uniformly in the region $x \geq c$.

Solution Say $|a_n| \leq M$. If $x \geq c > 1$, then

$$\left| \sum \frac{a_n}{n^x} \right| \leq M \sum \frac{1}{n^c}.$$

Because $c > 1$, this last series converges so by the Weierstrass M-Test, the original series converges uniformly for $x \geq c > 1$.

B–4. Let $\gamma(t)$ define a smooth curve that does not pass through the origin. If the point $\mathbf{P} = \gamma(t_0)$ is a point on the curve that is closest to the origin (and *not* an end point of the curve), show that the position vector $\gamma(t_0)$ is perpendicular to the tangent vector $\gamma'(t_0)$.

Solution At t_0 the function $h(t) := \|\gamma(t)\|^2 = \langle\gamma(t),\, \gamma(t)\rangle$ has a local minimum. Thus

$$0 = h'(t_0) = 2\langle\gamma(t_0),\, \gamma'(t_0)\rangle,$$

so $\gamma(t_0)$ is orthogonal to $\gamma'(t_0)$. Now observe that $\gamma'(t_0)$ is just the tangent vector at t_0.

B–5. Let A be an $n\times n$ real symmetric matrix with eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ and corresponding orthonormal eigenvectors $v_1, \ldots, v_n$. Show that

$$\lambda_2 = \min_{x\neq 0,\ x\perp v_1} \frac{\langle x,\, Ax\rangle}{\|x\|^2}.$$

Solution Since the eigenvectors form an orthonormal basis, we can write $x = x_1v_1+\cdots+x_nv_n$ for some x_j. Then $\langle x,\, Ax\rangle = \lambda_1x_1^2 + \lambda_2x_2^2 + \cdots + \lambda_nx_n^2$. Now $x \perp v_1$ implies that $x_1 = 0$. Since $\lambda_2 \le \cdots \le \lambda_n$ we find

$$\langle x,\, Ax\rangle \ge \lambda_2(x_2^2 + \cdots + x_n^2) = \lambda_2\|x\|^2.$$

Equality occurs if $x = v_2$.

B–6. Let $f(x)$ be a continuous function for $0 \le x \le 1$. Compute $J_n(f) := \lim_{n\to\infty} n\int_0^1 f(x)e^{-3nx}\,dx$.

Solution Observe that if $x \ge \delta > 0$, then $ne^{-3nx} \le ne^{-2n\delta} \to 0$, while at $x = 0$ the function $ne^{-3nx} = n$ blows-up. Thus all the action occurs at $x = 0$.

METHOD 1: Write

$$\begin{aligned} J_n(f) =& n\int_0^1 [f(x) - f(0)]e^{-3nx}\,dx + n\int_0^1 f(0)e^{-3nx}\,dx \\ =& n\int_0^1 [f(x) - f(0)]e^{-3nx}\,dx + \frac{1}{3}f(0). \end{aligned}$$

I show that for n large the first integral in the above line can be made arbitrarily small. Given $\epsilon > 0$, pick δ do that if $0 \le x < \delta$ then $|f(x) - f(0)| < \epsilon$. Also, say $|f(x| \le M$ for $0 \le x \le 1$. Then

$$\begin{aligned} n\int_0^1 [f(x) - f(0)]e^{-3nx}\,dx &\le \int_0^\delta + \int_\delta^1 \\ &\le \epsilon n\int_0^\delta e^{-3nx}\,dx + 2Mn\int_\delta^1 e^{-3nx}\,dx \le \frac{1}{3}\epsilon + \frac{1}{3}2Me^{-3n\delta}. \end{aligned}$$

Now let $n \to \infty$

METHOD 2: If f is smooth you can integrate by parts to compute the limit. More precisely, if $h \in C^1[0,1]$ then

$$\begin{aligned} J_n(h) &= -\frac{1}{3}h(x)e^{-3nx}\Big|_0^1 + \frac{1}{3}\int_0^1 h'(x)e^{-3nx}\,dx \\ &= \frac{1}{3}\left[-h(1)e^{-3n} + h(0) + \int_0^1 h'(x)e^{-3nx}\,dx\right]. \end{aligned}$$

Because $h \in C^1[0,1]$, h' is bounded: $|h'| \le M$. Therefore $\left|\int_0^1 h'(x)e^{-3nx}\,dx\right| < M/3n$. Consequently

$$|J_n(h) - \tfrac{1}{3}h(0)| < \frac{1}{3}\left[-h(1)e^{-3n} + \frac{M}{3n}\right] \to 0.$$

If f is only continuous, use the Weierstrass approximation theorem to find a smooth function h so that $|f(x) - h(x)| < \epsilon$ for all $x \in [0,1]$. Then write

$$J_n(f) - \tfrac{1}{3}f(0) = J_n(f-h) + [J_n(h) - \tfrac{1}{3}h(0)] + \tfrac{1}{3}[h(0) - f(0)].$$

Since $|f(x) - h(x)| < \epsilon$, then $|J_n(f-h)| < \epsilon/3$ and $\frac{1}{3}[h(0) - f(0)] < \epsilon/3$. Letting $n \to \infty$ we also have $J_n(h) - \frac{1}{3}h(0) \to 0$. Thus $J_n(f) \to f(0)/3$.

B–7. Let $\mathcal{D}$ be a bounded region in the plane, and $\mathcal{B}$ be its boundary (assumed smooth). Let $u(x,y,t)$ be a solution of the heat equation $\dfrac{\partial u}{\partial t} = \Delta u$ for (x,y) in $\mathcal{D}$. Say that the temperature $u(x,y,t) = 0$ for all points (x,y) on the boundary $\mathcal{B}$.

If $E(t) := \frac{1}{2}\iint_{\mathcal{D}} u^2(x,y,t)\,dx\,dy$, show that $dE/dt \le 0$.

Solution

$$\frac{dE}{dt} = \iint_{\mathcal{D}} uu_t\,dx\,dy = \iint_{\mathcal{D}} u\Delta u\,dx\,dy.$$

Now integrate by parts (the divergence theorem) to get

$$\iint_{\mathcal{D}} u\Delta u\,dx\,dy = \int_{\mathcal{B}} u\nabla u \cdot N\,ds - \iint_{\mathcal{D}} |\nabla u|^2\,dx\,dy,$$

where N is the unit outer normal and ds is the element of arc length, respectively, on $\mathcal{B}$. Since $u = 0$ on the boundary, the integral over $\mathcal{B}$ is zero so

$$\frac{dE}{dt} = -\iint_{\mathcal{D}} |\nabla u|^2\,dx\,dy \le 0.$$