1) p. 196 #3

$$u(x_0, y_0, z_0) = \frac{z_0}{2\pi}\iint \left[(x-x_0)^2 + (y-y_0)^2 + z_0^2\right]^{-3/2} h(x,y)\,dx\,dy$$

let $x' = x - x_0$ $\quad dx' = dx$

$y' = y - y_0$ $\quad dy' = dy$

$$= \frac{z_0}{2\pi}\iint \left(x'^2 + y'^2 + z_0^2\right)^{-3/2} h(x'+x_0, y'+y_0)\,dx'\,dy'$$

change to cylindrical: $r^2 = x'^2 + y'^2$

$dx'dy' = r\,dr\,d\theta$

$$= \frac{z_0}{2\pi}\int_0^{2\pi}\int_0^{\infty} (r^2 + z_0^2)^{-3/2} h(x'+x_0, y'+y_0)\, r\,dr\,d\theta$$

let $s = \frac{r}{z_0}$ $\quad ds = \frac{1}{z_0}dr$

$$= \frac{z_0}{2\pi}\int_0^{2\pi}\int_0^{\infty} \frac{1}{z_0^3}(s^2+1)^{-3/2} h(x'+x_0, y'+y_0)\, s\, z_0^2\, ds\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} (s^2+1)^{-3/2} s\, h(z_0 s\cos\theta + x_0, z_0 s\sin\theta + y_0)\, ds\, d\theta$$

as $x' = r\cos\theta = s z_0 \cos\theta$

$y' = r\sin\theta$

send $z_0 \to 0$

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} (s^2+1)^{-3/2} s\, h(x_0, y_0)\, ds\, d\theta$$

$= h(x_0, y_0)$ ✓

3) p. 196 # 6

a) $G(x, y, x_0, y_0) = -\frac{1}{2\pi} \log |\vec{x} - \vec{x}_0| + \frac{1}{2\pi} \log |\vec{x} - \vec{x}_0^*|$

$$= \frac{1}{2\pi}\left[\log\left((x-x_0)^2+(y+y_0)^2\right)^{1/2} - \log\left((x-x_0)^2+(y-y_0)^2\right)^{1/2}\right]$$

where $\vec{x}_0 = (x_0, y_0)$

$\vec{x}_0^* = (x_0, -y_0)$.

b) $\frac{\partial G}{\partial n} = -\frac{\partial G}{\partial y}\Big|_{y=0} = \frac{1}{2\pi}\left[\frac{1}{|\vec{x}-\vec{x}_0^*|} \cdot \frac{1}{2} |\vec{x}-\vec{x}_0^*|^{-1} \cdot 2(y+y_0)\right.$

$$\left. - \frac{1}{|\vec{x}-\vec{x}_0|} \cdot \frac{1}{2} |\vec{x}-\vec{x}_0^*|^{-1} \cdot 2(y-y_0)\right]$$

$$= \frac{1}{\pi} \frac{y_0}{|\vec{x}-\vec{x}_0|^2}$$

$$\Rightarrow u(x_0, y_0) = \frac{y_0}{\pi}\int_{-\infty}^{\infty} \frac{h(x)}{[(x-x_0)^2 + (y_0)^2]}\, dx$$

c) $h(x) = 1$

$$u(x_0, y_0) = \frac{y_0}{\pi}\int_{-\infty}^{\infty} \frac{1}{[(x-x_0)^2 + (y_0)^2]}\, dx$$

Let $s = \frac{x - x_0}{y_0}$

$ds = \frac{1}{y_0} dx$

$$= \frac{\cancel{y_0}}{\pi}\int \frac{1}{\cancel{y_0^2}(s^2+1)} \cdot \cancel{y_0}\, ds$$

$$= \frac{1}{\pi}\int \frac{1}{1+s^2}\, ds$$

$$= \frac{1}{\pi} \arctan(s) \Big|_{-\infty}^{\infty} = \frac{1}{\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \underline{1}$$

4) p. 197 #11 $\qquad |x - x_0| = \rho$ and $|x - x_0^*| = \rho^*$

a) $G(x, x_0) = \frac{1}{2\pi}\log\rho - \frac{1}{2\pi}\log\left(\frac{r_0}{a}\rho^*\right)$

i) $G \in C^2$ and $\Delta G = 0$ except at $|x - x_0^*|$ ✓. as log functions

ii) $G(x, x_0) = 0$ for $|x| = a$.

pf. $|x| = a \Rightarrow x_0^* = \frac{a^2 x_0}{|x_0|^2} = x_0 \Rightarrow \rho = \rho^*$

$$G(x, x_0) = \frac{1}{2\pi}\log\rho - \frac{1}{2\pi}\log\left(\frac{a}{a}\rho\right) = 0 \quad ✓$$

iii) No singularity at

$G(x, x_0) - \frac{1}{2\pi}\log\rho$ $\quad$ as $\quad \rho^* \neq 0$ anywhere in the circle.

b)

$$\nabla\rho = \frac{x - x_0}{\rho} \quad \text{as} \quad 2\rho\nabla\rho = 2(x - x_0)$$

$$\nabla\rho^* = \frac{x - x_0^*}{\rho^*}$$

$$\Rightarrow \nabla G = \frac{x - x_0}{2\pi\rho^2} - \frac{x - x_0^*}{2\pi\rho^{*2}} = \frac{1}{2\pi\rho^2}\left[x - x_0 - \frac{r_0^2}{a^2}x + x_0\right]$$

$$= \frac{a^2 - r_0^2}{2\pi a^2\rho^2} \quad \text{on } |x| = a.$$

$$\Rightarrow \frac{\partial G}{\partial n} = \frac{x}{a}\cdot\nabla G = \frac{a^2 - r_0^2}{2\pi a\rho^2}.$$

$$\Rightarrow u(x_0) = \int_{|x|=a} u(x)\frac{\partial G}{\partial n}\,ds = \frac{a^2 - |x_0|^2}{2\pi a}\int_{|x|=a}\frac{h(x)}{|x - x_0|^2}\,ds$$

5) p. 197 #13

Find Green's Function for half ball $z>0$.

$$G(x,x_0) = -\frac{1}{4\pi\rho} + \frac{a}{r_0}\frac{1}{4\pi\rho^*} \quad \text{for full ball.}$$

Need to reflect across plane.

Let $\tilde{\rho} = |\vec{x} - \vec{\tilde{x}}_0|$ where $\tilde{x}_0 = (x_0, y_0, -z_0)$

$\tilde{\rho}^*$ is reflection across ball and z-plane.

We need $G(x,x_0)=0$ on bd D which includes ball and $z=0$ plane.

$$\text{So } G(x,x_0) = -\frac{1}{4\pi\rho} + \frac{a}{r_0}\frac{1}{4\pi\rho^*} + \frac{1}{4\pi\tilde{\rho}} - \frac{a}{r_0}\frac{1}{4\pi\tilde{\rho}^*}$$

6) p. 233 #1

$$u(x,t) = f(k\cdot\vec{x} - ct) = f(k_1x + k_2y + k_3z - ct)$$

$$u_{tt} = c^2\Delta u = c^2(u_{xx} + u_{yy} + u_{zz})$$

$$c^2 f''(k\cdot\vec{x} - ct) = c^2\left(k_1^2 f''(\quad) + k_2^2 f''(\quad) + k_3^2 f''(\quad)\right)$$

$$\Rightarrow f''(k\cdot\vec{x} - ct) = |k|\, f''(k\cdot\vec{x} - ct)$$

$$\Rightarrow |k| = 1$$

7) p. 233 #2

Undefined function on light cone.

Otherwise,

Let $r^2 = x^2+y^2+z^2$

$u(r,t) = (c^2t^2 - r^2)^{-1}$

$\Delta u = u_{rr} + \frac{2}{r}u_r$ as $u_\theta = u_\varphi = 0$.

$$u_{tt} = \frac{2c^2(3c^2t^2+r^2)}{(c^2t^2-r^2)^3}$$

$$u_r = \frac{2r}{(c^2t^2-r^2)^2}$$

$$u_{rr} = \frac{2(3r^2+c^2t^2)}{(c^2t^2-r^2)^3}$$

$$\frac{2}{r}u_r + u_{rr} = \frac{2(r^2+3c^2t^2)}{(c^2t^2-r^2)^3} = u_{tt} \checkmark$$

8) p. 233 #6

As in book,

$$E'(t) = \frac{1}{2}\iiint_D (u_t^2 + c^2|\nabla u|^2)_t \, dx = c^2 \iiint_D (\nabla u_t \cdot \nabla u + u_t \Delta u)\, dx$$

$$= c^2 \iint_{\partial D} u_t \frac{\partial u}{\partial n}\, dS = 0$$

in either bd. condition

$(u_t = 0$ or $\frac{\partial u}{\partial n} = 0)$.