1) a) $e^x = \underbrace{\left(\frac{e^x + e^{-x}}{2}\right)}_{\text{even}} + \underbrace{\left(\frac{e^x - e^{-x}}{2}\right)}_{\text{odd}}$

b) $f(x) = \varphi(x) + \psi(x)$

$\varphi(x) = \varphi(-x)$ $\qquad\qquad \psi(x) = -\psi(-x)$

$f(-x) = \varphi(x) - \psi(x)$

$\Rightarrow f(x) + f(-x) = 2\varphi(x)$

$\Rightarrow \varphi(x) = \dfrac{f(x) + f(-x)}{2}$

$f(x) - f(-x) = 2\psi(x)$

$\Rightarrow \psi(x) = \dfrac{f(x) - f(-x)}{2}$

2) $f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$

$g(x) = \begin{cases} 0 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$

f: $A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\left(\frac{n\pi x}{\ell}\right)dx = 0$ as f is odd.

$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$

$= \frac{2}{\pi}\int_{0}^{\pi}\sin(nx)\,dx$ as f is even and $f \equiv 1$ on $0 < x < \pi$

g: $A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{0}^{\pi}\cos(nx)\,dx$

$= \frac{1}{n\pi}\sin(nx)\Big|_0^{\pi}$

$= \frac{1}{n\pi}\left[\sin(n\pi) - 0\right]$

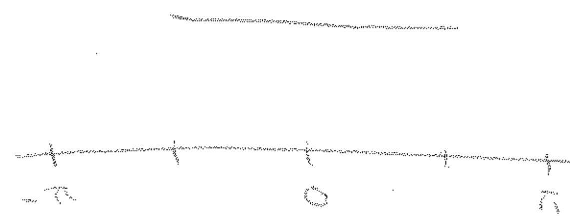
$= 0$ except $A_0 = 1$

$B_n = \frac{1}{\pi}\int_0^{\pi}\sin(nx)\,dx$

$\Rightarrow A_n^f = A_n^g = 0$ except for $A_0^f = 0$ and $A_0^g = 1$

$B_n^f = 2B_n^g$

$$h(x) = \begin{cases} 0 & -\pi < x < -\pi/2 \\ 1 & -\pi/2 < x < \pi/2 \\ 0 & \pi/2 < x < \pi \end{cases}$$



$$A_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos(nx)\,dx$$

$A_0 = 1$ otherwise $= \dfrac{2}{\pi}\displaystyle\int_0^{\pi/2} \cos(nx)\,dx$

$$= +\frac{2}{n\pi}\sin(nx)\Big|_0^{\pi/2}$$

$$= \frac{2}{n\pi}\sin(n\pi/2) = \begin{cases} 0 & n \text{ even} \\ \frac{2}{n\pi}(-1)^m & n \text{ odd where } n = 2m+1 \end{cases}$$

$$B_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sin(nx)\,dx = 0 \quad \text{as odd.}$$

so $A_n^f = B_n^g = 0$

$$B_n^g = \frac{2}{\pi}\int_0^{\pi} \sin(nx)\,dx = \frac{2}{n\pi}\cos(nx)\Big|_0^{\pi}$$

$$= \frac{-2}{n\pi}\left[\cos(n\pi) - 1\right]$$

$$= \frac{-2}{n\pi}\left[(-1)^n - 1\right]$$

$$= \begin{cases} 0 & n \text{ even} \\ \frac{4}{n\pi} & n \text{ odd} \end{cases}$$

3) $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$

$u'' + \alpha u = f(x)$

a) Find $u(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$

$$u'(x) = \sum i n c_n e^{inx}$$

$$u''(x) = \sum -n^2 c_n e^{inx}$$

$$u'' + \alpha u = \sum \left(-c_n n^2 + \alpha c_n\right) e^{inx} = \sum a_n e^{inx}$$

$$a_n = c_n(\alpha - n^2)$$

$$c_n = \frac{a_n}{\alpha - n^2}$$

b) Difficulty for α any perfect square.

4) $u_{tt} = c^2 u_{xx}$

$-\pi \le x \le \pi$

$u(\pm\pi, t) = 0$ $\quad$ $u(x,0) = 1 - \frac{|x|}{\pi}$

$u_t(x,0) = 0$

Translate to $0, L$

then by 4.1 (9)

$$u(x,t) = \sum_n \left(A_n \cos\left(\frac{n\pi c t}{L}\right) + B_n \sin\left(\frac{n\pi c t}{L}\right)\right) \sin\left(\frac{n\pi x}{L}\right)$$

$$u(x,0) = 1 - \frac{|x - L/2|}{L/2} = \varphi(x)$$

$$u(x,0) = \sum_n \left(A_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

by 5.1 (4) $\Rightarrow A_0 = 0$

$$A_n = \frac{2}{L}\int_0^L \varphi(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L}\int_0^{L/2} \left(1 + \left(\frac{x - L/2}{L/2}\right)\right) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$+ \frac{2}{L}\int_{L/2}^{L} \left(1 - \left(\frac{x - L/2}{L/2}\right)\right) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{-4}{n^2\pi^2}\left(\sin(\pi n) - 2\sin\left(\frac{1}{2}\pi n\right)\right)$$

$$0 = u_t(x,0) = \sum_n B_n\left(\frac{n\pi c}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

$$B_n\left(\frac{n\pi c}{L}\right) = \int_0^L 0 \sin\left(\frac{m\pi x}{\ell}\right) = 0$$

$$\Rightarrow B_n = 0.$$

II

5) a) $|c_n| < \frac{Q}{n^2}$

$$\sum_{n=-\infty}^{\infty} |c_n e^{inx}| < Q \sum_{n=-\infty}^{\infty} \frac{1}{n^2}$$

$$= Q\left(c_0 + 2\sum_{n=1}^{\infty} \frac{1}{n^2}\right) < \infty$$

$\therefore$ series converges absolutely.

b) $$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$

$$|c_n| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) e^{-inx}|\, dx$$

$$= \frac{M}{2\pi}\int_{-\pi}^{\pi} dx = M$$

c) $f \in C^1[-\pi, \pi]$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \qquad f'(x) = \sum in c_n e^{inx}$$

$$\gamma_n = in c_n$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\, dx \qquad \gamma_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x) e^{-inx}\, dx$$

let $u = e^{-inx}$, $dv = f'(x)dx$

$du = -in e^{-inx}$, $v = f(x) dx$

$$\gamma_n = \frac{1}{2\pi}\left[-in e^{-inx} f(x) + in \int e^{-inx} f(x)\, dx\right]$$

$$= \frac{-in}{2\pi} e^{-inx} f(x) \Big|_{-\pi}^{\pi} + in c_n$$

$$\boxed{\gamma_n = in c_n}$$

6) a) $W = \sum_{k=1}^{N} c_k V_k + Z$

$\|w\|^2 = |c_1|^2\|v_1\|^2 + \dots + |c_N|^2\|v_N\|^2 + \|z\|^2$

$\|w\|^2 = \langle w, w \rangle = \left\langle \sum_{k=1}^{N} c_k v_k + z, \sum_{j=1}^{N} c_j v_j + z \right\rangle$ 6

$= \sum_{j=1}^{N}\sum_{k=1}^{N} \langle c_k v_k, c_j v_j \rangle + \cancel{\sum_{k=1}^{N} \langle c_k v_k, z \rangle}_0$

$+ \cancel{\sum_{k=1}^{N} \langle z, c_k v_k \rangle} + \langle z, z \rangle$

by linearity

$= \sum_{k=1}^{N} |c_k|^2 \langle v_k, v_k \rangle + \langle z, z \rangle$ by orthogonality.

b) $f = \sum_{-\infty}^{\infty} c_k e^{ikx}$

Show $\frac{1}{2\pi}\|f\|^2 \geq \sum_{k=-N}^{N} |c_k|^2$

Fix N,

$\cancel{0 \leq \| f - \sum_{-N}^{N} \langle f, e_k \rangle e_k \|^2}$

$\cancel{\|f\|^2 = \langle f, f \rangle = \sum}$

$f_N(x) = \sum_{k=-N}^{N} c_k e^{ikx}$

$$\langle f - f_N, f - f_N \rangle = \langle f, f \rangle - 2\langle f, f_N \rangle + \langle f_N, f_N \rangle$$

$$\| f - f_N \|^2$$

$$= \|f\|^2 - 2\langle f, f_N \rangle + \|f_N\|^2$$

$$\langle f_n, f_n \rangle = \int_{-\pi}^{\pi} |f_n|^2 \, dx$$

$$= \int_{-\pi}^{\pi} \left| \sum_{-N}^{N} c_k e^{ikx} \right|^2 dx$$

$$= \int_{-\pi}^{\pi} \sum |c_k|^2 \, dx \quad \text{by orthogonality.}$$

$$= 2\pi \sum_{-N}^{N} |c_k|^2$$

$$\langle f, f_N \rangle = \int_{-\pi}^{\pi} f(x) \overline{f_N(x)} \, dx$$

7) $\varphi(x) \in C^0(0,\ell)$

$\tilde{\varphi}(x)$ odd extension

$$\tilde{\varphi}(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\left(\frac{n\pi x}{\ell}\right) + B_n \sin\left(\frac{n\pi x}{\ell}\right)\right)$$

$$A_n = \frac{1}{\ell}\int_{-\ell}^{\ell} \tilde{\varphi}(x)\cos\left(\frac{n\pi x}{\ell}\right)dx$$

$$= 0 \quad \text{as } \tilde{\varphi}\cdot\cos \text{ odd.}$$

$$B_n = \frac{1}{\ell}\int_{-\ell}^{\ell} \tilde{\varphi}(x)\sin\left(\frac{n\pi x}{\ell}\right)$$

$$= \frac{2}{\ell}\int_{0}^{\ell} \tilde{\varphi}(x)\sin\left(\frac{n\pi x}{\ell}\right) \quad \text{as } \tilde{\varphi}\cdot\sin \text{ even}$$

$$= \frac{2}{\ell}\int_{0}^{\ell} \varphi(x)\sin\left(\frac{n\pi x}{\ell}\right)$$

$$\Rightarrow \tilde{\varphi}(x) = \underbrace{\sum_{n=1}^{\infty}\left[\frac{2}{\ell}\int_0^{\ell}\varphi(x)\sin\left(\frac{n\pi x}{\ell}\right)dx \ \sin\left(\frac{n\pi x}{\ell}\right)\right]}_{\text{which is sine series.}}$$

8) a) f even

$f(x) = f(-x)$

$f'(x) = f'(-x) = -f'(-x)$ by chain rule.

$\therefore f'$ odd.

Similar for f odd.

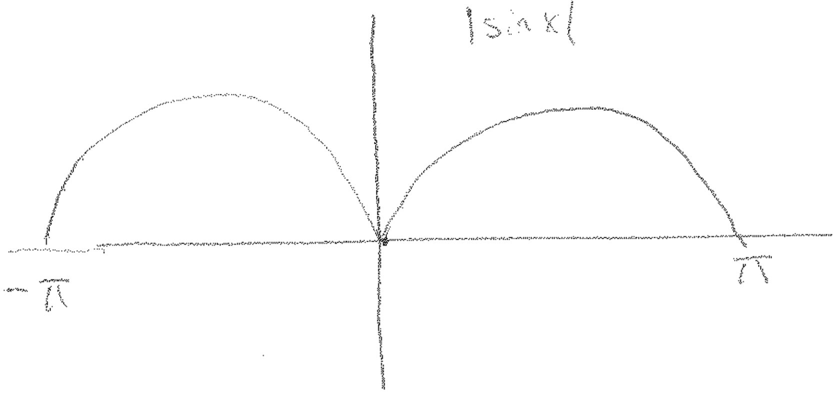
b) let $g(x) = \int_0^x f(s)\,ds$

chose 0 as we don't care about constant

$g(-x) = \int_0^{-x} f(s)\,ds$ let $t = -s$, $dt = -ds$

$= -\int_0^{x} f(t)\,dt$

9) $|\sin x|$ on $(-\pi, \pi)$



$|\sin x|$ is even

$$B_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\sin(x)| \sin\left(\frac{n\pi x}{\ell}\right) dx = 0 \quad \forall n$$

as odd.

$$A_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\sin(x)| \cos\left(\frac{n\pi x}{\ell}\right) dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \sin(x) \cos\left(\frac{n\pi x}{2\pi}\right) dx$$

All B_n's disappear!