1) $u'' + b(t)u' + c(t)u = 0$

on $[0,A]$

$|b(t)|, |c(t)| \leq M$

a) $E(t) := \frac{1}{2}(u'^2 + u^2) \geq uu'$ (by $2xy \leq x^2+y^2$)

$E'(t) = \frac{1}{2}(2u'u'' + 2uu')$

$= u'(-b(t)u' - c(t)u) + uu'$

$= -b(t)u'^2 - c(t)uu' + uu'$

$\leq |b(t)|(u')^2 + |c(t)|uu' + uu'$

$\leq M(u')^2 + (1+M)uu'$

$\leq M(u')^2 + (1+M)E(t)$

$\leq 2ME(t) + (1+M)E(t)$ (as $Mu'^2 \leq Mu'^2 + Mu^2 = 2ME(t)$)

$= \gamma E(t)$ where $\gamma = 3M+1$

b) $(e^{-\gamma t}E(t))' = -\gamma e^{-\gamma t}E(t) + e^{-\gamma t}E'(t)$

$= e^{-\gamma t}(-\gamma E(t) + E'(t)) \leq 0$ by (a) $\forall t \in [0,A]$

$\therefore f(t) = e^{-\gamma t}E(t)$ has maximum at $f(0) = E(0)$.

$\Rightarrow e^{-\gamma t}E(t) \leq E(0) \Rightarrow E(t) \leq e^{\gamma t}E(0)$

c) Let $u(0) = u'(0) = 0$

$\Rightarrow E(0) = 0$

$\Rightarrow 0 \leq E(t) \leq e^{\gamma t}E(0) = 0$

as $E(t)$ is sum of squares

$\Rightarrow E(t) = 0$

Hence as $E(t)$ is sum of squares, $u(t) = 0$ $\forall t \in [0,A]$

d) $u = v - w$

$$u'' + b(t)u' + c(t)u = v'' + b(t)v' + c(t)v - w'' - b(t)w' - c(t)w$$
$$= f(t) - f(t) = 0$$

$u(0) = u'(0) = 0$

By part c, $u \equiv 0 \Rightarrow v \equiv w$ in $[0, A]$

e) $\varphi(P) = \varphi(0)$

$\varphi'(P) = \varphi'(0)$

Let $\psi(t) = \varphi(t+P)$

Compare φ, ψ on $[0, P]$

$\psi(0) = \varphi(P) = \varphi(0)$

$\psi'(0) = \varphi'(P) = \varphi'(0)$

And

$$\psi''(t) + b(t)\psi'(t) + c(t)\psi(t) = \varphi''(t+P) + b(t)\varphi'(t+P) + c(t)\varphi(t+P)$$
$$= \varphi''(t+P) + b(t+P)\varphi'(t+P) + c(t+P)\varphi(t+P)$$
$$= f(t+P) \quad \text{as } \varphi \text{ is solution}$$
$$= f(t) \quad \text{as } f \text{ periodic.}$$

$\therefore$ by uniqueness, $\psi \equiv \varphi$ on $[0, P]$

$\Rightarrow \varphi(t) = \varphi(t+P)$

$\Rightarrow \varphi$ is periodic

$\square$

2) $u_t = u_{\theta\theta}$

$u(\theta+2\pi) = u(\theta)$

a) $E(t) = \frac{1}{2}\int_{-\pi}^{\pi} u^2(\theta,t)\,d\theta$

$$E'(t) = \int_{-\pi}^{\pi} u u_t \, d\theta$$

$U = u \quad dV = u_{\theta\theta}\,d\theta$

$dU = u_\theta d\theta \quad V = u_\theta$

$$= \int_{-\pi}^{\pi} u u_{\theta\theta}\,d\theta = +\cancel{u u_\theta \Big|_{-\pi}^{\pi}} - \int_{-\pi}^{\pi} u_\theta^2 \, d\theta$$

0 as periodic

$$\leq 0$$

b) $u(\theta, 0) = 0$

$\Rightarrow E(0) = 0$

$\Rightarrow E(t) \equiv 0 \quad \forall t \geq 0 \quad$ as $E'(t) \leq 0$ and $E(t) \geq 0$

$\Rightarrow u(\theta, t) = 0 \quad \forall t \geq 0$

3) $u_t = u_{xx}$ on $(-L, L)$

$u(-L,t) = u(L,t) = 0$

$u(x,0) = f(x)$.

a) $E'(t) = \int_{-L}^{L} u u_t \, dx = \int_{-L}^{L} u u_{xx} = \cancel{u u_x \Big|_{-L}^{L}} - \int_{-L}^{L} u_x^2 \, dx \leq 0$

0 as periodic

b) let v be another solution,

$w = u - v$

$w(-L,t) = w(L,t) = 0$.

$\therefore w(x,t) \equiv 0$ identical to 2b).

c) If $u(x,t)$ even, then $u(x,t) = u(-x,t) \Rightarrow u(x,t) - u(-x,t) = 0$

$w(x,t) = u(x,t) - u(-x,t)$ solves $w_t = w_{xx}$ as $w_t = u_t(x,t) - u_t(-x,t)$

$w_{xx} = u_{xx}''(x,t) - u_{xx}''(-x,t)$

and $w(x,0) = \varphi(x) - \varphi(-x) = 0$ as φ even

$\therefore$ by uniqueness, $w \equiv 0$.

4) $u_{xx} + u_{xt} - 20u_{tt} = 0$

$u(x,0) = \varphi(x)$

$u_t(x,0) = \psi(x)$

$$\left(\frac{\partial}{\partial x} + 5\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial t}\right) u = 0$$

$$\Rightarrow u(x,t) = f(5x-t) + g(4x+t)$$

$$\varphi(x) = f(5x) + g(4x)$$

$$\psi(x) = -f'(5x) + g'(4x)$$

rewrite $u(x,t) = f(x - \frac{1}{5}t) + g(x + \frac{1}{4}t)$

$$\varphi(x) = f(x) + g(x)$$

$$\psi(x) = -\frac{1}{5}f'(x) + \frac{1}{4}g'(x)$$

$$\left.\begin{aligned} \varphi' &= f' + g' \\ \psi &= -\tfrac{1}{5}f' + \tfrac{1}{4}g' \end{aligned}\right\} f' = \varphi' - g'$$

$$\psi = \frac{1}{5}\varphi' - \frac{1}{5}g' + \frac{1}{4}g'$$

$$= \frac{1}{5}\varphi' + \frac{1}{20}g'$$

$$\Rightarrow g' = 20\psi - 4\varphi'$$

$$\Rightarrow f' = -20\psi + 5\varphi'$$

$$\Rightarrow f(s) = 5\varphi(s) - 20\int_0^s \psi + A$$

$$g(s) = -4\varphi(s) + 20\int_0^s \psi + B$$

$$\varphi(s) = f(s) + g(s) = \varphi(s) + A + B \Rightarrow A + B = 0$$

$$\Rightarrow u(x,t) = 5\varphi(x - \tfrac{1}{5}t) - 20\int_0^{x-\frac{1}{5}t} \psi - 4\varphi(x + \tfrac{1}{4}t) + 20\int_0^{x+\frac{1}{4}t} \psi$$

$$= 5\varphi(x - \tfrac{1}{5}t) - 4\varphi(x + \tfrac{1}{4}t) + 20\int_{x-\frac{1}{5}t}^{x+\frac{1}{4}t} \psi(s)\, ds$$

$\square$

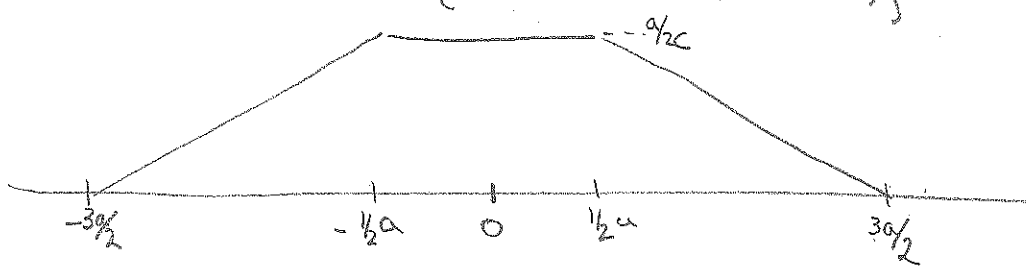
5) $u(x,0) = 0$

$u_t(x,0) = g(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| \geq a \end{cases}$

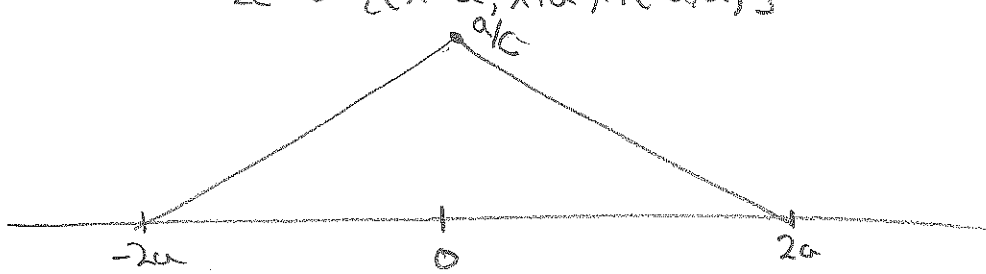
$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds = \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$

$= \frac{1}{2c} \cdot \text{Length}\{(x-ct, x+ct) \cap (-a,a)\}$

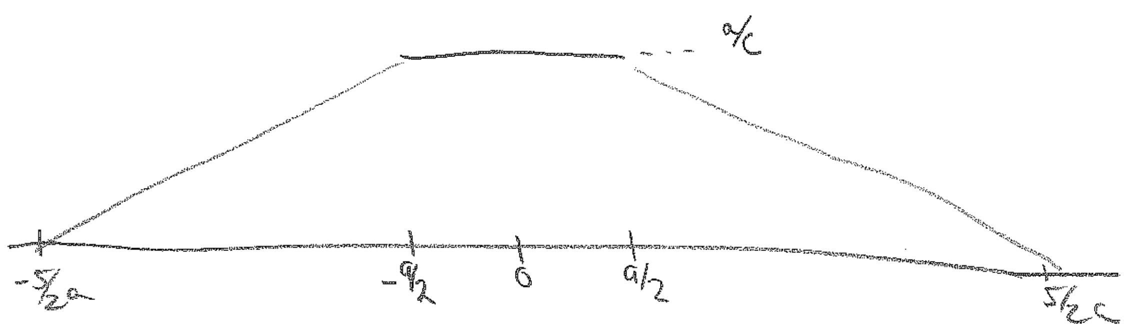
$t = \frac{1}{2}a/c \Rightarrow u(x,t) = \frac{1}{2c}\text{Length}\{(x-\frac{1}{2}a, x+\frac{1}{2}a) \cap (-a,a)\}$



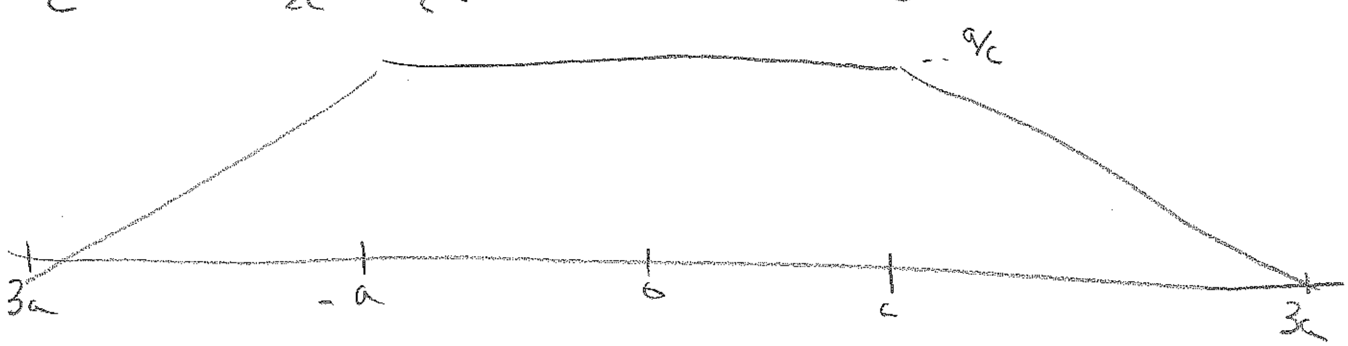
$t = a/c \Rightarrow u(x,t) = \frac{1}{2c}\text{Length}\{(x-a, x+a) \cap (-a,a)\}$



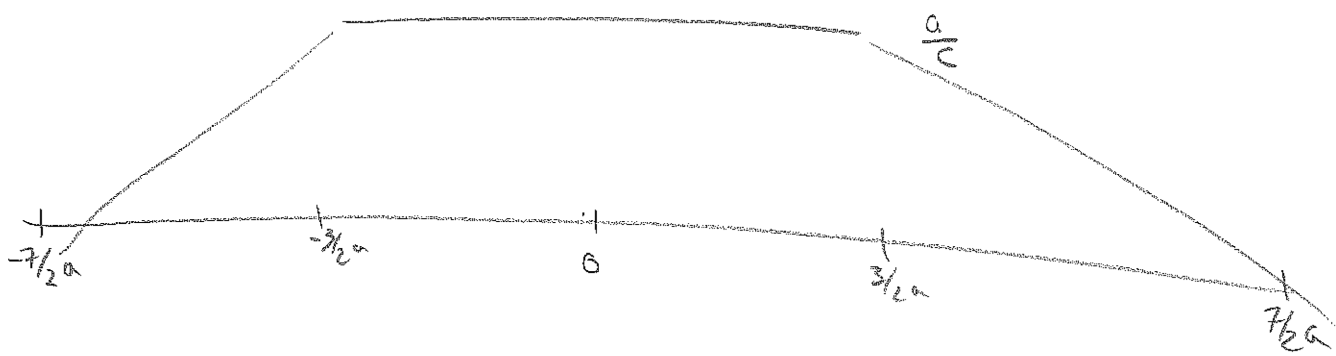
$t = \frac{3a}{2c} \Rightarrow u(x,t) = \frac{1}{2c}\text{Length}\{(x-\frac{3}{2}a, x+\frac{3}{2}a) \cap (-a,a)\}$



$t = \frac{2a}{c} \Rightarrow u(x,t) = \frac{1}{2c}\text{Length}\{(x-2a, x+2a) \cap (-a,a)\}$



$t = \frac{5}{2}\frac{a}{c}$ $\quad u(x,t) = \frac{1}{2c}\,\text{Length}\left\{\left(x - \frac{5}{2}a,\ x + \frac{5}{2}a\right) \cap (-a, a)\right\}$



6) $u_{tt} = c^2 u_{xx} \quad x \geq 0$

$u(x,0) = 3 - \sin x \ = \varphi(x)$

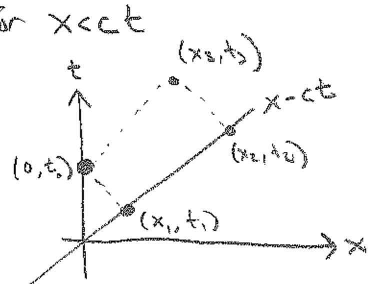
$u_t(x,0) = 0 \quad = \psi(x)$

$u(0,t) = 3 - t^2$

For $x > ct$ (away from $u(0,t)$)

$u(x,t) = \frac{1}{2}\left(3 - \sin(x+ct) + 3 - \sin(x-ct)\right)$ by D'Alembert

For $x < ct$



$x_3 - ct_3 = 0 - ct_0$

$t_0 = t_3 - \frac{x_3}{c}$

From previous HW,

$$u(x_3,t_3) + u(x_1,t_1) = u(x_2,t_2) + u(0,t_0)$$

$$= u(x_2,t_2) + u\left(0, t_3 - \frac{x_3}{c}\right)$$

$$= u(x_2,t_2) + 3 - \left(t_3 - \frac{x_3}{c}\right)^2$$

$$\Rightarrow u(x_3,t_3) = u(x_2,t_2) - u(x_1,t_1) + 3 - \left(t_3 - \frac{x_3}{c}\right)^2$$

$$= \frac{1}{2}\left[\cancel{3} - \sin(x_2+ct_2) + \cancel{3} - \sin(x_2-ct_2)\right] - \frac{1}{2}\left[\cancel{3} - \sin(x_1+ct_1) + 3 - \sin(x_1-ct_1)\right]$$
$$+ 3 - \left(t_3 - \frac{x_3}{c}\right)^2$$

$x_1 + ct_1 = ct_0 = ct_3 - x_3$

$x_2 + ct_2 = ct_3 + x_3$

$$\Rightarrow \quad u(x,t) = \begin{cases} 3 - \frac{1}{2}\left[\sin(x+ct) + \sin(x-ct)\right] & x \geq ct \\ 3 - \left(t - \frac{x}{c}\right)^2 - \frac{1}{2}\left[\sin(x+ct) + \sin(x-ct)\right] & x < ct \end{cases}$$

$\square$

7) $\varphi(x) = e^{-x}$

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left[e^{-(x-y)^2/4kt} - e^{-(x+y)^2/4kt} \right] e^{-y}\, dy$$

8) $$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^\infty e^{-(x-y)^2/4kt} f(y)\, dy$$

a) v solves diffusion $(v_t = v_{xx})$ on the whole line with $v(x,0) = f(x)$

b) $w = v_x - 2v$

w satisfies diffusion, as

$W_t = V_{xt} - 2V_t$

$W_{xx} = (V_{xxx}) - 2V_{xx}$

$= V_{tx} - 2V_t = W_t$

$w(x,0) = v_x(x,0) - 2v(x,0)$

$= f'(x) - 2f(x)$

c) Show $f'(x) - 2f(x)$ is odd

Say $x > 0$

$f(x) = x$

$f'(x) = 1$

$\therefore f'(x) - 2f(x) = 1 - 2x$

$f(-x) = -x + 1 - e^{-2x}$

$f'(-x) = -1 + 2e^{-2x}$

$f'(-x) - 2f(-x) = \cancel{2e^{-2x}} - 2x + 2 - \cancel{2e^{-2x}} - 1$

$= -\left(f'(x) - 2f(x) \right)$

Hence, odd!

d) Did on previous HW, therefore, w is odd w.r.t. X

e) $\therefore$ $v(x,t)$ satisfies $v_t = k v_{xx}$

$v(x,0) = x$ for $x > \infty$

and $v_x(0,t) - 2v(0,t) = 0$ for $x = 0$ $\rightarrow$ w is odd so $w(0) = 0$.

$\Rightarrow$ by uniqueness,

this is our solution to Robin boundary problem

9) $u_{tt} = u_{xx}$ $\quad 0 < x < \infty$

$$\left.\begin{aligned} u(x,0) &= \varphi(x) \\ u_t(x,0) &= \psi(x) \end{aligned}\right\} \text{ for } x \geq 0$$

$u_x(0,t) = 0$

Want u even.

So define φ_{ev}, ψ_{ev} even extensions of u

i.e. $\varphi_e(-x) = \varphi_e(x) = \varphi(x)$

$\therefore$ u is even

$u(x) = u(-x)$

$u'(x) = -u'(-x)$ so u is odd in x.

$\Rightarrow u_x(0,t) = 0$

$$u(x,t) = \frac{1}{2}[\varphi_e(x+ct) + \varphi_e(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_e(y)\,dy$$

For $x > c|t|$

$$= \frac{1}{2}[\varphi(x+ct) + \varphi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy$$

let $s = -y$
$ds = -dy$
$y=0$ $s=0$
$y=x-ct$ $s=ct-x$

For $x < ct$

$$= \frac{1}{2}[\varphi(x+ct) + \varphi(ct-x)] + \frac{1}{2c}\int_0^{x+ct} \psi(y)\,dy + \frac{1}{2c}\int_{x-ct}^{0} \psi(-y)\,dy$$

$$= \frac{1}{2}[\varphi(x+ct) + \varphi(ct-x)] + \frac{1}{2c}\int_0^{x+ct} \psi(y)\,dy + \frac{1}{2c}\int_0^{ct-x} \psi(y)\,dy$$

10) p. 66 #3

$f(x+ct)$ for $t<0$

Find vibrations if the end $x=0$ is fixed. For $t>0$

For $t<0$ $\quad u(x,t) = f(x+ct)$ $\qquad u_{tt} - c^2 u_{xx} = 0$

$u(0,t) = 0$

$u(x,0) = f(x)$

$u_t(x,0) = c f'(x)$

for $x > ct$

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2}\int_{x-ct}^{x+ct} f'(y)\,dy$$

$$= \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2}\left[f(x+ct) - f(x-ct)\right]$$

$$= f(x+ct)$$

for $x < ct$

$$u(x,t) = \frac{1}{2}\left[f(ct+x) - f(ct-x)\right] + \frac{1}{2}\left[f(ct+x) - f(ct-x)\right]$$

$$= f(x+ct) - f(ct-x)$$