1) $u'' - c(x)u = 0 \qquad c(x) > 0$

a) If u has local max, $u''(x_0) < 1$

$\therefore \quad u''(x_0) - \underset{\text{pos}}{\underset{\uparrow}{c(x_0)}}\, \underset{\text{pos}}{\underset{\uparrow}{u(x_0)}} < 0$ which violates solution.

b)

$\underset{\text{pos}}{\underset{\downarrow}{u''(x_0)}} - \underset{\text{pos}}{\underset{\updownarrow}{c(x_0)}}\, \underset{\text{neg}}{\underset{\downarrow}{u(x_0)}} > 0$ which violates solution.

c) If $u(0) = u(1) = 0$, then $u(x) \equiv 0$ as we cannot have pos. max or neg. min.

2) p. 31 #1

(a) $u_{xx} \underbrace{- u_{xy} + 2u_y + u_{yy} - 3u_{yx}}_{} + 4u = 0$

$= -4u_{xy}$

$\Rightarrow a_{12} = -2 \qquad a_{11} = 1 \quad a_{22} = 1 \qquad a_1 = 0 \quad a_2 = 2$

$a_0 = 4$

$a_{12}^2 = 4 > a_{11} a_{22}$

$\boxed{\text{Hyperbolic}}$

(b) $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0$

$a_{12} = 3$

$a_{12}^2 = 9 = a_{11}a_{22} = (9)(1)$

$\boxed{\text{Parabolic}}$

3) p.31 #5

$$u_{xx}+3u_{yy}-2u_x+24u_y+5u=0$$

$$u=ve^{\alpha x+\beta y}$$

$$u_x=v_xe^{\alpha x+\beta y}+\alpha ve^{\alpha x+\beta y}$$

$$u_{xx}=v_{xx}e+\alpha v_xe+\alpha v_xe+\alpha^2ve$$

$$u_y=v_ye+\beta ve$$

$$u_{yy}=v_{yy}e+\beta v_ye+\beta v_ye+\beta^2ve$$

$$v_{xx}+2\alpha v_x+\alpha^2v+3v_{yy}+6\beta v_y+3\beta^2v-2v_x-2\alpha v$$
$$+24v_y+24\beta v+5v=0$$

$$\Rightarrow v_{xx}+(2\alpha-2)v_x+3v_{yy}+(6\beta+24)v_y$$
$$+(\alpha^2+3\beta^2-2\alpha+24\beta+5)v$$

set $\alpha=1$, $\beta=-4$

$$\Rightarrow v_{xx}+3v_{yy}+(-44)v=0$$

$v(x',y')=v(x,\gamma y)$

$v_{yy}(x',y')=\gamma^2v(x,y)$

$\boxed{\text{so } \gamma=\sqrt{3}}$

$$\Rightarrow v_{xx}+v_{y'y'}-44v=0$$

4) p. 38 # 8

$$u_{tt} = c^2\left(u_{rr} + \frac{2}{r}u_r\right)$$

(a) $v = ru \quad \Rightarrow \quad u = \frac{1}{r}v$

$v_r = u + ru_r$

$$u_r = -\frac{1}{r^2}v + \frac{1}{r}v_r$$

$$u_{rr} = \frac{2}{r^3}v - \frac{1}{r^2}v_r - \frac{1}{r^2}v_r + \frac{1}{r}v_{rr}$$

$$u_t = \frac{1}{r}v_t$$

$$u_{tt} = \frac{1}{r}v_{tt}$$

$$\frac{1}{r}v_{tt} = c^2\left(\cancel{\frac{2}{r^3}v} - \cancel{\frac{2}{r^2}v_r} + \frac{1}{r}v_{rr} + \cancel{\frac{2}{r}\left(-\frac{1}{r^2}v + \frac{1}{r}v_r\right)}\right)$$

$$\frac{1}{r}v_{tt} = c^2\left(\frac{1}{r}v_{rr}\right)$$

$$v_{tt} = c^2 v_{rr}$$

(b) $v_{tt} - c^2 v_{rr}$

$$v(r,t) = f(r+ct) + g(r-ct)$$

(c) $u(r,0) = \varphi(r)$, $u_t(r,0) = \psi(r)$ } even $\qquad v(r,0) = r\varphi(r)$, $v_t(r,0) = r\psi(r)$

$$r\varphi(r) = v(r,0) = f(r) + g(r)$$

$$r\psi(r) = cf'(r) - cg'(r)$$

5) p. 38 #11

$$3u_{tt} + 10u_{xt} + 3u_{xx} = \sin(x+t)$$

$$\left(3\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + 3\frac{\partial}{\partial x}\right)u = \sin(x+t)$$

So homogeneous solution is of form

$$u(x,t) = f(3t-x) + g(t-3x)$$

Particular solution

$$u = C\sin(x+t)$$

$$u_t = +C\cos(x+t)$$

$$u_{tt} = -C\sin(x+t) \qquad u_{xx} = -C\sin(x+t)$$

$$u_{xt} = -C\sin(x+t)$$

$$3u_{tt} + 10u_{xt} + 3u_{xx} = -16C\sin(x+t)$$

$$\Rightarrow C = -\frac{1}{16}$$

$$u(x,t) = -\frac{1}{16}\sin(x+t) + f(3t-x) + g(t-3x)$$

6) p. 41 #4

$$u_{tt} = u_{xx}$$

$u(x+h,\ t+k) + u(x-h,\ t-k) =$

$$\frac{1}{2}\left[\varphi(x+h+c(t+k)) + \varphi(x+h-c(t+k))\right] + \frac{1}{2c}\int_{(x+h)-c(t+k)}^{(x+h)+c(t+k)} \psi(s)\,ds$$

$$+ \frac{1}{2}\left[\ -\quad -\quad -\quad -\ \right] + \frac{1}{2c}\int_{-\ -}^{-\ -}$$

Regroup! $\boxed{c=1}$

$$= \frac{1}{2}\left[\varphi(x+k+t+h) + \varphi(x+k-(t+h))\right] + \frac{1}{2}\int_{(x-k)-(t-h)}$$

or more generally,

$$u(x,t) = f(x+t) + g(x-t)$$

$$u(x+h,t+k) = \overset{A}{f(x+h+t+k)} + \overset{B}{g(x+h-t-k)}$$

$$+\ u(x-h,t-k) = \overset{C}{f(x-h+t-k)} + \overset{D}{g(x-h-t+k)}$$

$$A+B+C+D = \underbrace{\overset{A}{f((x+k)+(t+h))} + \overset{D}{g((x+k)-(t+h))}}_{u(x+k,\,t+h)} + \underbrace{\overset{C}{f(x-k+t-h)} + \overset{B}{g(x-k-(t-h))}}_{u(x-k,\,t-h)}$$

7) p. 45 #1

$1-x^2-2kt = u(x,t)$

$\{0 \le x \le 1,\ 0 \le t \le T\}$

$u_x = -2x \qquad u_t = -2k$

No interior pts as $u_t \ne 0$.

$\boxed{x=0}$

$u(0,t) = 1-2kt = f(t)$

$f'(t) = -2k \ne 0$

$\boxed{x=1}$

$u(1,t) = -2kt$

$f'(t) = -2k \ne 0$

$\boxed{t=0}$

$u(x,0) = 1-x^2 = f(x)$

$f'(x) = -2x$

$(0,0)$

$\boxed{t=T}$

$u(x,T) = 1-x^2-2kT$

$f'(x) = -2x$

$(0,T)$

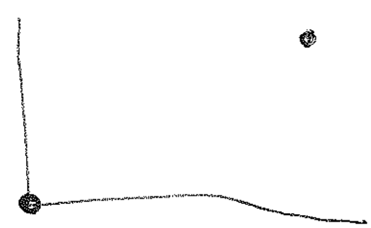
Check Corners,

$(0,0) \to 1$ Max.

$(0,T) \to 1-2kT$

$(1,0) \to 0$

$(1,T) = -2kT$ Min.



8) p. 46 #4

$u_t = u_{xx}$ in $\{0<x<1,\ 0<t<\infty\}$

with $u(0,t) = u(1,t) = 0$

$u(x,0) = 4x(1-x)$.

Know max occurs either for $t=0$ or $x=0, x=1$

$u(0,t) = u(1,t) = \boxed{0}$.

min

$u(x,0) = 4x(1-x)$

$f(x) = 4x(1-x)$

$f'(x) = 4(1-x) - 4x$

$= 4 - 8x \quad \boxed{x = 1/2}$

$u(\tfrac{1}{2},0) = 2(1/2) = 1$ so $\boxed{0 < u(x,t) < 1}$

b) $u(x,t) = u(1-x,t)$ as

$v(x,t) = u(1-x,t)$

$v(0,t) = u(1,t) = 0$

$v(1,t) = u(0,t) = 0$

$v(x,0) = u(1-x,0) = 4(1-x)(1-1+x)$

$= 4x(1-x)$.

$v_t = u_t$

$v_{xx} = u_{xx}$

by uniqueness, $u(x,t) = u(1-x,t)$

c) $\int_0^1 \frac{d}{dt} u^2\, dx = \int_0^1 2uu_t\, dx = \int_0^1 2u\, u_{xx}\, dx$

$U = u \quad dV = u_{xx}\,dx$

$dU = u_x\,dx \quad V = u_x$

$= \overset{0}{u u_x \Big|_0^1} - 2\int_0^1 u_x^2\, dx < 0$

9) a) $V_t - V_{xx} = M \quad V(0,t) = A \quad V(\cancel{a},\cancel{t}) = B$

V doesn't depend on t!

$\cancel{V(x,t) = \tfrac{1}{2}M}$ $\qquad V_t = 0$

$V_{xx} = -M$

$V_x = -Mx + C_1$

$V = -\frac{1}{2}Mx^2 + C_1 x + C_2$

$V(0,t) = C_2 = a$

$V(x,t) = -\frac{1}{2}Mx^2 + C_1 x + a$

$V(L,t) = -\frac{1}{2}ML^2 + C_1 L + a = b$

$C_1 L = b - a + \frac{1}{2}ML^2$

$C_1 = \dfrac{b - a + \frac{1}{2}ML^2}{L}$

$$\boxed{V(x,t) = -\frac{1}{2}Mx^2 + \left(\frac{b - a + \frac{1}{2}ML^2}{L}\right)x + a}$$

b) $u \le v$.

10) p. 52 #1

$\varphi(x) = 1 \quad |x| < l \qquad \varphi(x) = 0 \text{ for } |x| > l$

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-l}^{l} e^{-(x-y)^2/4kt}\, dy$$

let $p = \dfrac{(x-y)}{2\sqrt{kt}} \qquad dy = -2\sqrt{kt}\, dp$

$2\sqrt{kt}\, p = (x-y) \qquad y = -l \rightarrow p = \dfrac{(x+l)}{2\sqrt{kt}}$

$y = x - 2\sqrt{kt}\, p \qquad y = l \rightarrow p = \dfrac{(x-l)}{2\sqrt{kt}}$

$$u(x,t) = \frac{-\cancel{2\sqrt{kt}}}{\sqrt{\cancel{4}\pi \cancel{kt}}} \int_{\frac{(x+l)}{2\sqrt{kt}}}^{\frac{(x-l)}{2\sqrt{kt}}} e^{-p^2}\, dp$$

$$= \cancel{\frac{1}{\sqrt{\pi}}\,\mathrm{Erf}} + \frac{1}{\sqrt{\pi}} \int_{\frac{(x-l)}{2\sqrt{kt}}}^{\frac{(x+l)}{2\sqrt{kt}}} e^{-p^2}\, dp$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\frac{(x+l)}{2\sqrt{kt}}} - \int_0^{\frac{(x-l)}{2\sqrt{kt}}}$$

$$= \frac{1}{2}\,\mathrm{Erf}\left(\frac{x+l}{2\sqrt{kt}}\right) - \frac{1}{2}\,\mathrm{Erf}\left(\frac{x-l}{2\sqrt{kt}}\right)$$

11) p. 53 #11

$u(x,0) = \varphi(x)$.

$\varphi(-x) = -\varphi(x) \quad \forall x$.

$u(-x,t) + u(x,t) = v(x,t)$.

$$V_t = u_t(-x,t) + u_t(x,t)$$
$$V_x = -u_x(-x,t) + u_x(x,t)$$
$$V_{xx} = u_{xx}(-x,t) + u_{xx}(x,t)$$

$$V_t = V_{xx} \quad \checkmark.$$

$$V(x,0) = u(-x,0) + u(x,0)$$
$$= \varphi(-x) + \varphi(x)$$
$$= -\varphi(x) + \varphi(x) = 0.$$

$\therefore$ by uniqueness,

since $V_t = V_{xx}$ and $V(x,0) = 0$

and $V(x,t) = 0$ is a solution,

$$V(x,t) = 0. \checkmark$$

b) $u(x,t) - u(-x,t)$

12) p.54 #16

$$u_t - ku_{xx} + bu = 0 \qquad -\infty < x < \infty$$

$$u(x,0) = \varphi(x)$$

$$b > 0.$$

$$u(x,t) = v(x,t)e^{-bt}$$

$$u_t = -bve^{-bt} + v_t e^{-bt}$$

$$u_{xx} = v_{xx}e^{-bt}$$

$$-bve^{-bt} + v_t e^{-bt} - kv_{xx}e^{-bt} + bve^{-bt} = 0$$

$$v_t - kv_{xx} = 0$$

$$v(x,0) = u(x,0)e^{b(t=0)} = \varphi(x)$$

$$\Rightarrow v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \varphi(y)\,dy$$

$$\Rightarrow u(x,t) = \frac{e^{-bt}}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \varphi(y)\,dy.$$