1) $u'' = f(x)$

$-1 \le x \le 1$

$u(-1) = u(1) = 0$

$$u'(x) = b + \int_{-1}^{x} f(t)\,dt$$

$$u(x) = a + bx + \int_{-1}^{x}\int_{-1}^{t} f(s)\,ds\,dt$$

$$= a + bx + \int_{-1}^{x} (x-t) f(t)\,dt$$

$$u(-1) = a - b + 0$$

$$\Rightarrow a = b \quad \omega \quad u(-1) = 0.$$

$$u(1) = 2a + \int_{-1}^{1} (1-t) f(t)\,dt = 0.$$

$$\Rightarrow a = -\tfrac{1}{2}\int_{-1}^{1} (1-t) f(t)\,dt$$

$$\Rightarrow u(x) = -\tfrac{1}{2}(1+x)\int_{-1}^{1} (1-t) f(t)\,dt + \int_{-1}^{x} (x-t) f(t)\,dt$$

$$= \left(\int_{-1}^{x} + \int_{x}^{1}\right)(-\tfrac{1}{2})(1+x)(1-t) f(t)\,dt + \int_{-1}^{x} (x-t) f(t)\,dt$$

$$= \int_{-1}^{x}\left[(-\tfrac{1}{2})(1+x)(1-t) + (x-t)\right] f(t)\,dt + \int_{x}^{1} (-\tfrac{1}{2})(1+x)(1-t) f(t)\,dt$$

$$= \int_{-1}^{x} (-\tfrac{1}{2})(1-x)(1+t)\, f(t) + \int_{x}^{1} -\tfrac{1}{2}(1+x)(1-t) f(t)\,dt.$$

$$\Rightarrow G(x,t) = \begin{cases} -\tfrac{1}{2}(1-x)(1+t) & -1 \le t \le x \\ -\tfrac{1}{2}(1+x)(1-t) & x \le t \le 1. \end{cases}$$

2)

First show $\overline{(u)}_r = \overline{(u_r)}$

$$\overline{(u)}_r = \frac{\partial}{\partial r}\frac{1}{4\pi r^2}\iint_{|x|=r} u\, dS = \frac{\partial}{\partial r}\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u \sin\theta \, d\theta \, d\varphi$$

$$= \frac{1}{4\pi}\iint u_r \sin\theta \, d\theta \, d\varphi$$

$$= \overline{(u_r)} \quad \checkmark$$

Now, $(\overline{\Delta u}) = \frac{1}{4\pi r^2}\iint_{|x|=r} \Delta u \, dS$

$$= \frac{1}{4\pi r^2}\iint \underbrace{u_{rr} + \frac{1}{r}u_r}_{\substack{\text{already}\\ \text{good here!}}} + \underbrace{\frac{1}{r^2}\left(u_{\theta\theta} + \cot\theta \cdot u_\theta + \frac{1}{\sin^2\theta} u_{\varphi\varphi}\right)}_{1} dS$$

so ~~just~~ look at other half.

$$1 = \int_0^{\pi}\int_0^{2\pi} \sin\theta\, u_{\theta\theta} + \cos\theta\, u_\theta + \frac{1}{\sin\theta} u_{\varphi\varphi} \, d\varphi \, d\theta$$

$$= 2\pi\int_0^{\pi} (u_\theta \sin\theta)_\theta \, d\theta + \int_0^{\pi} \frac{1}{\sin\theta}\left[u_\varphi\Big|_0^{2\pi}\right] d\theta$$

$$= 2\pi\left[u_\theta \cancel{\sin\theta}^{\,0}\Big|_0^{\pi} + 0\right] = 0.$$

$\therefore$ these terms go to 0

$$\therefore \quad \Delta\bar{u} = \overline{\Delta u}$$

3)

$$u(x_0,t_0) = \frac{1}{4\pi c^2 t_0} \iint_S \psi(\vec{x})\, d\vec{x}$$

$$= \frac{1}{4\pi c^2 t_0} \int_0^{2\pi}\int_0^{\pi} (y_0 + ct_0 \sin\theta \cos\varphi) \sin\theta\, c^2 t_0^2 \, d\theta\, d\varphi$$

$$= \frac{t_0}{4\pi} \int_0^{2\pi}\int_0^{\pi} (y_0 + ct_0 \sin\theta\cos\varphi) \sin\theta \, d\theta\, d\varphi$$

$= t_0 \cdot$ avg value of sphere centred at (x_0, y_0, z_0)

$= t_0 y_0$.

4) Vanishes for $|\vec{x} - \vec{x}_0| > ct_0$.

5) a) $\vec{x} = (x_0, y_0, z_0)$ $R_0 = |\vec{x}_0|$

$$\Theta = \arccos \frac{R^2 + R_0^2 - \rho^2}{2RR_0}$$

$$\Rightarrow A = R^2 \int_0^{2\pi}\int_0^{\pi} \sin\theta\, d\theta\, d\varphi = 2\pi R^2 \int_0^{\Theta} \sin\theta\, d\theta$$

$$= 2\pi R^2 \left(1 - \frac{R^2 + R_0^2 - \rho^2}{2RR_0}\right)$$

$$= 2\pi R^2 - \pi R \left(\frac{R^2 + R_0^2 - \rho^2}{R_0}\right)$$

b) For $ct_0 < \rho$

$$u(x_0, t_0) = \begin{cases} A t_0 & |x_0| < \rho - ct_0 \\ \frac{A t_0}{2}\left(1 - \frac{c^2 t_0^2 + |x_0|^2 - \rho^2}{2ct_0 |x_0|}\right) & \rho - ct_0 < |x_0| < \rho + ct_0 \end{cases} \quad \text{(from a)}$$

For $ct_0 > \rho$

$$u(x_0, t_0) = \frac{At_0}{2}\left(1 - \frac{c^2t_0^2 + |x_0|^2 - \rho^2}{2ct_0|x_0|}\right).$$

e)

$$tu(x_0, tv, t) = \frac{At^2}{2}\left(1 - \frac{c^2t^2 + |x_0 + tv|^2 - \rho^2}{2ct|x_0 + tv|}\right)$$

$$= \frac{A}{2}\left(\frac{\rho^2 t - t(|x_0+tv| - ct)^2}{2c|x_0+tv|}\right)$$

let $t \to \infty$

Note

$$|x_0+tv| - ct = \frac{\frac{1}{t}|x_0|^2 + 2\langle x_0, v\rangle}{|\frac{x_0}{t} + v| + c}$$

$\therefore$ as $t \to \infty$

$$\to A \frac{\rho^2 - \frac{\langle x_0, v\rangle^2}{c^2}}{4c|\frac{x_0}{t} + v|} < \infty$$

$\square$

6) let $v = ru$

then $\Delta u = u_{tt}$

becomes $v_{tt} = c^2 v_{rr}$.

This is 1D wave equation w/ solutions

$v = f(r+ct) + g(r-ct)$

$\Rightarrow u = \frac{1}{r} f(r+ct) + \frac{1}{r} g(r-ct)$

7) IC $\Rightarrow$ $F(x) + G(x) = f(x)$

$H(x) + K(x)$

We need continuity.

$u(0^+, t) = u(0^-, t)$

$u_x(0^+, t) = u_x(0^-, t)$

⋮

algebra...