LECTURES ON PARTIAL DIFFERENTIAL EQUATIONS

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trigonometric series, the series (7, 38) with these coefficients converges absolutely and uniformly to $\phi^*(x)$. Since for $t \geqslant 0$

$$0 < \exp\left\{-\frac{k^2\pi^2}{l^2}t\right\} \leqslant 1,$$

the series (6, 38) also converges absolutely and uniformly for $t \geqslant 0$. Consequently, the function $u^*(t, x)$ defined by this series is continuous in the rectangle Q: $0 \leqslant x \leqslant l$, $0 \leqslant t \leqslant T$, and takes on the prescribed values on its lower base and vertical sides. It remains to show that $u^*(t, x)$ satisfies the heat equation in the interior of Q and on its upper base. For this it suffices to show that the series obtained by differentiating (6, 38) termwise once with respect to t and twice with respect to x converge absolutely and uniformly. This follows from the fact that for every positive t

$$\frac{k^2\pi^2}{l^2}\exp\left\{-\frac{k^2\pi^2}{l^2}t\right\} < 1,$$

provided k is sufficiently large.

It is possible to show in the same way as above that the function u^* has continuous derivatives of all orders with respect to x and t in the interior of Q and on its upper base. Using this fact it is easy to show that, if we keep the original homogeneous conditions at $x=0$ and $x=l$, it is impossible, in general, to extend the just constructed function $u^*(t, x)$ in the direction of the negative t's so that it satisfies equation (1, 38). Indeed, if such extension were possible, we would obtain a solution of the heat equation in a rectangle Q_1

$$0 < x < l,$$
$$-\epsilon < t \leqslant 0,$$

and this solution would vanish at $x=0$ and $x=l$. Applying to Q_1 the same considerations which we applied before to Q, we find that $u^*(0, x)$, i.e. $\phi^*(x)$, must have derivatives of all orders.

Even if $u^*(0, x) = \phi^*(x)$ were such that it would be possible to solve the first boundary-value problem in Q_1 with $u^*(0, x) = \phi^*(x)$ and with homogeneous conditions at the end-points of the interval $(0, l)$, we could alter this solution by as much as we please for arbitrarily small negative values of t by changing by as little as we please the function $\phi^*(x)$ and any of its derivatives up to some arbitrary fixed order k. For this, as is easily verified, it suffices to

add to the original solution a term of the series (6, 38) with sufficiently large subscript and with an arbitrarily small constant multiplier. Consequently, if the initial conditions refer to $t=0$, the first boundary value problem is reasonable for $t>0$ and is not reasonable for $t<0$ (cf. § 8). This again demonstrates the disparity between positive and negative values of t in the case of the heat equation (1, 38).

§ 39. Conduction of heat in an infinite strip (the Cauchy problem)

1. *Statement of the problem.* We wish to determine for $t \geqslant 0$ a continuous and bounded function of t and x which satisfies the heat equation (1, 38) for $t>0$ and which, for $t=0$, becomes equal to a given continuous and bounded function $\phi(x)$, defined for all real x.

2. *Proof of the uniqueness of the solution.* It is clear that to prove uniqueness it suffices to show that, for $\phi(x) \equiv 0$, the solution of the problem just stated is identically equal to zero. We denote by $u(t, x)$ some bounded solution of the latter problem whose absolute values do not exceed M, and we show that $u \equiv 0$. To this end we construct an auxiliary function $u_1(t, x)$ defined by the following conditions:

(*a*) $u_1(t, x)$ must satisfy the heat equation in the interior of the square

$$|x| \leqslant 1,$$
$$0 \leqslant t \leqslant 1;$$

(*b*) $u_1(t, x)$ must be continuous in this square and satisfy the following conditions on its boundary:

$$u_1(t, 0) = M, \quad u_1(t, l) = M \quad \text{for} \quad 0 \leqslant t \leqslant 1,$$
$$u_1(0, x) = 0 \quad \text{for} \quad |x| \leqslant 1 - \epsilon, \ \epsilon > 0;$$

(*c*) for $t=0$, $u_1(t, x)$ must be linear in the intervals

$$(-1, -1+\epsilon), \quad (1-\epsilon, 1).$$

The existence of such function was proved in the preceding section. It follows from the maximum theorem that $u_1(t, x) \geqslant 0$ everywhere.

Let us put

$$u_l(t, x) = u_1\left(\frac{t}{l^2}, \frac{x}{l}\right).$$

It is clear that for every positive l the function $u_l(t, x)$ satisfies the heat equation in the interior of the rectangle

$$|x| \leqslant l, \quad 0 \leqslant t \leqslant l^2,$$

that it is equal to M on its vertical sides, that it vanishes on the segment $t=0$, $|x| \leqslant l(1-\epsilon)$, and that $u_l(0, x)$ is linear in the intervals $(-l, -l(1-\epsilon))$ and $(l(1-\epsilon), l)$. Since the function $u_1(t, x)$ vanishes at the point $t=x=0$ and is continuous in its neighborhood, all values of $u_1(t/l^2, x/l)$, l sufficiently large, are positive and arbitrarily small in any fixed rectangle Q^*,

$$|x| \leqslant X; \quad 0 \leqslant t \leqslant T,$$

On the other hand, it is easily verified that, for arbitrary sufficiently large l, a solution $u(t, x)$ of our problem satisfies the inequality

$$|u(t,x)| \leqslant u_l(t,x) \tag{1, 39}$$

on the lower base and vertical sides of the rectangle Q^*. Consequently, the same inequality must hold in the interior of Q^*. However, according to what has just been said, if (t, x) is an arbitrary but fixed point of the square Q^* with $t>0$, the right side of the inequality (1, 39) is arbitrarily small for l sufficiently large, and the left side of (1, 39) is independent of l. Hence $u(t, x)=0$ at an arbitrary point (t, x), $t>0$. This completes the proof of uniqueness.

A. N. Tikhonov showed that the solution of the problem considered may not be unique in the class of unbounded functions and found a uniqueness condition.†

3. *Continuous dependence of the bounded solution on the initial function* $\phi(x)$. To prove this continuous dependence it suffices to show that $|\phi(x)| \leqslant \epsilon$ implies $|u(t, x)| \leqslant \epsilon$. This is proved in a manner analogous to that of the uniqueness proof just given, with the auxiliary function $u_1(t, x)$ replaced by the function $u_1(t, x)+\epsilon$.

4. We prove that the solution of our problem is given by the formulas

$$u(t,x) = \frac{1}{2\pi^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{\phi(\xi)}{t^{\frac{1}{2}}} \exp\left\{-\frac{(x-\xi)^2}{4t}\right\} d\xi \quad \text{for} \quad t>0, \tag{2, 39}$$

$$u(0,x) = \phi(x). \tag{3, 39}$$

The integral (2, 39) is known as the *Poisson integral.*

† *Matematichesky sbornik*, vol. XLII, 2 (1935), pp. 199–216 (in Russian).

It is easy to check that the integral (2, 39) converges for all positive t. It is just as easy to show that the integrals obtained by differentiating (2, 39) under the integral sign any number of times with respect to t and x converge. Also, all these integrals converge uniformly in the neighborhood of (t, x) $(t>0)$. It follows that, for $t>0$, $u(t, x)$, defined by (2, 39), and its derivatives of all orders with respect to x and t exist. Since the integrand satisfies equation (1, 38) for $t>0$, it follows that $u(t, x)$ itself satisfies equation (1, 38) for $t>0$.

We now show that the function $u(t, x)$ defined by formula (2, 39) is bounded for $t>0$. We first observe that, if we put

$$M_1 = \max_{-\infty<x<\infty} |\phi(x)|,$$

then

$$|u(t,x)| \leqslant \frac{1}{2\pi^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{M_1}{t^{\frac{1}{2}}} \exp\left\{-\frac{(x-\xi)^2}{4t}\right\} d\xi = \frac{M_1}{\pi^{\frac{1}{2}}} \int_{-\infty}^{\infty} \exp\{-\eta^2\} d\eta = M_1.$$

It remains to show that the function $u(t, x)$ is continuous for $t=0$, i.e. for every x_0

$$\left|\frac{1}{2\pi^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{\phi(\xi)}{t^{\frac{1}{2}}} \exp\left\{-\frac{(x-\xi)^2}{4t}\right\} d\xi - \phi(x_0)\right| < \epsilon, \tag{4, 39}$$

provided t and $|x-x_0|$ are sufficiently small. We observe that to prove the continuity of $u(t, x)$ for $t=0$ it suffices to prove that for small t

$$\left|\frac{1}{2(\pi t)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \phi(\xi) \exp\left\{-\frac{(x-\xi)^2}{4t}\right\} d\xi - \phi(x)\right| < \tfrac{1}{2}\epsilon. \tag{5, 39}$$

That this is so follows from the fact that, in view of the continuity of the function $\phi(x)$, $|\phi(x)-\phi(x_0)|$ is small if $|x-x_0|$ is sufficiently small.

To prove the relation (5, 39) we rewrite the Poisson equation in the form

$$\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} \phi(x+2t^{\frac{1}{2}}\zeta) \exp\{-\zeta^2\} d\zeta$$

by putting $\xi = x + 2t^{\frac{1}{2}}\zeta$, and observe that

$$\phi(x) = \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} \phi(x) \exp\{-\zeta^2\} d\zeta.$$

In view of the boundedness of $\phi(x)$, the absolute values of the integrals

$$\int_{-\infty}^{-N} \phi(x+2t^{\frac{1}{2}}\zeta) \exp\{-\zeta^2\} d\zeta, \quad \int_{-\infty}^{-N} \phi(x) \exp\{-\zeta^2\} d\zeta,$$

$$\int_{N}^{\infty} \phi(x+2t^{\frac{1}{2}}\zeta) \exp\{-\zeta^2\} d\zeta, \quad \int_{N}^{\infty} \phi(x) \exp\{-\zeta^2\} d\zeta$$

are as small as we please for N sufficiently large. Consequently, for N sufficiently large, we have, approximately,

$$u(t,x) = \pi^{-\frac{1}{2}} \int_{-N}^{N} \phi(x + 2t^{\frac{1}{2}}\zeta) \exp\{-\zeta^2\}\, d\zeta,$$

and

$$\phi(x) = \pi^{-\frac{1}{2}} \int_{-N}^{N} \phi(x) \exp\{-\zeta^2\}\, d\zeta,$$

and the approximation is as good as we please. But for t sufficiently small, the right sides of these approximate equalities are, by the continuity of $\phi(x)$, arbitrarily close to each other. This implies the correctness of (5, 39).

5. We have thus shown that the unique solution of the problem stated in the beginning of this section is that defined by equations (2, 39) and (3, 39).

These formulas imply in particular that, if $\phi(x)$ is positive on an arbitrarily small x interval and otherwise equal to zero everywhere, then $u(t,x)$ is positive for all values of x and for an arbitrary fixed $t > 0$. This implies the paradoxical result that heat is conducted in a strip with infinite speed. While this is physically impossible, it nevertheless follows inevitably from the assumption that equation (1, 38) provides an exact description of the conduction of heat in a strip.

Clearly, the hypotheses on which our derivation of this equation is based are not quite justified by experience.

§ 40. Survey of some further investigations of equations of the parabolic type

1. Existence and uniqueness of the solution of the first boundary-value problem for the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \dots + \frac{\partial^2 u}{\partial x_n^2} \qquad (1, 40)$$

have been proved for arbitrary n. In the simplest case this problem can be stated as follows:

We seek a continuous function $u(t, x_1, \dots, x_n)$ defined on the closure of a region G bounded ('from below' and 'from above') by pieces of the planes $t = 0$ and $t = T$ and on the sides by one or a few surfaces with continuously turning tangent planes nowhere parallel to the t-axis. The function u must satisfy equation (1, 40) in the interior of G and must coincide with some function f, prescribed

on the boundary of G with the exception of its upper base. This function is supposed continuous on the closed set on which it is defined. The proofs of the uniqueness of the solution of this problem and of its continuous dependence on the function f are carried out in exactly the same way as in § 38.

2. Results analogous to those stated in para. 1 have been proved for the case when the condition $u = f$ on the boundary is replaced by the condition $(\partial u / \partial n) + \sigma u = f$ on that boundary.

3. The results stated in paras. 1 and 2 have also been proved for the solutions of the parabolic equation

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} A_{ij}(t, x_1, \dots, x_n) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} B_i(t, x_1, \dots, x_n) \frac{\partial u}{\partial x_i}$$
$$+ C(t, x_1, \dots, x_n)\, u + D(t, x_1, \dots, x_n). \qquad (2, 40)$$

Here it is assumed that the form

$$\sum_{i,j=1}^{n} A_{ij}(t, x_1, \dots, x_n)\, \alpha_i \alpha_j$$

is positive-definite at every point $(t, x_1, \dots, x_n)$ of the region under consideration, and that the coefficients A, B, C, D have continuous derivatives of sufficiently high orders.†

4. If the coefficients of equation (2, 40) are analytic in all their arguments, then, as was shown by Gevrey,† all sufficiently smooth solutions of this equation are analytic in the arguments $(x_1, \dots, x_n)$ and have derivatives of all orders with respect to t, without being necessarily analytic in t.

5. If the initial values of u are prescribed in the whole hyperplane $t = 0$, then there exists a unique bounded solution of the Cauchy problem for equation (2, 40) (this problem is analogous to that considered in § 39). Here it is assumed that $u(0, x_1, \dots, x_n)$ is bounded and continuous.

† Gevrey, *J. Math.*, 6 série, vol. IX (1913), vol. X (1914); *Ann. Ecole Norm. sup.*, 3 série, vol. XXXV (1918).