Lorentz Transformations

Orthogonal Transformations

In Euclideal space $\mathbb{R}^2$ (and $\mathbb{R}^n$), it is valuable to find all the invertible linear changes of variable $y = Rx$ that preserve the length of a vector

$$\|Rx\| = \|x\|$$

so that

$$y_1^2 + y_2^2 = x_1^2 + x_2^2.$$

These are *orthogonal transformations*. Say

$$\begin{aligned} x_1 =& ay_1 + by_2 \\ x_2 =& cy_1 + dy_2. \end{aligned}$$

Then

$$x_1^2 + x_2^2 = (a^2 + c^2)y_1^2 + 2(ab + cd)y_1 y_2 + (b^2 + d^2)y_2^2.$$

We therefore want

$$a^2 + c^2 = 1, \qquad ab + cd = 0, \quad \text{and} \quad b^2 + d^2 = 1$$

There are four variables and only three conditions so we will have one free parameter. To satisfy the first condition it is natural to let $a = \cos\theta$ and $c = \sin\theta$. For the second condition, let $b = -c = -\sin\theta$ and $d = a = \cos\theta$. The third condition is also satisfied. This gives the matrix

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

These are the rotations of ther plane $\mathbb{R}^2$.

By a similar computation, these are also the only linear changes of variable that preserve the Laplace operator

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = \frac{\partial^2 u}{\partial y_1^2} + \frac{\partial^2 u}{\partial y_2^2}.$$

Lorentz Transformations

It is also valuable to find all linear changes of variable

$$\begin{aligned} x' &= \alpha x + \beta t \\ t' &= \gamma x + \delta t \end{aligned}$$

that preserve the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t'^2} - c^2 \frac{\partial^2 u}{\partial x'^2},$$

where c is a constant (the speed of sound or light).

By the chain rule,

$$u_{tt} - c^2 u_{xx} = (\delta^2 - c^2\gamma^2) u_{t't'} + 2(\beta\delta - c^2\alpha\gamma) u_{x't'} + (\beta^2 - c^2\alpha^2) u_{x'x'}.$$

Thus we want

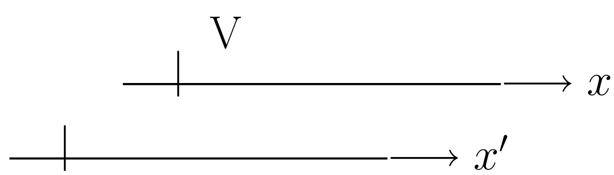
$$\delta^2 - c^2\gamma^2 = 1, \qquad \beta\delta - c^2\alpha\gamma = 0, \quad \text{and} \quad \beta^2 - c^2\alpha^2 = -c^2$$

First pick γ and δ so that $\delta^2 - c^2\gamma^2 = 1$, and then let $\beta = \pm c^2\gamma$, $\alpha = \pm\delta$. To preserve orientation we use the + signs. Since $c^2\alpha^2 - \beta^2 = c^2$ and $\cosh^2\sigma - \sinh^2\sigma = 1$, it is traditional to write $\alpha = \cosh\sigma$, $\beta = c \sinh\sigma$. For any real σ the transformation

$$\begin{aligned} x' &= (\cosh\sigma)\, x + (c \sinh\sigma)\, t \\ t' &= (\frac{1}{c} \sinh\sigma)\, x + (\cosh\sigma)\, t \end{aligned} \tag{1}$$

preserves the wave operator. This is called a *Lorentz transformation*. Lorentz [1853–1928] transformations also preserve arc length $ds^2 := dx'^{\,2} - c^2\,dt'^{\,2} = dx^2 - c^2\,dt^2$ in space-time and are fundamental in the study of the wave operator and special relativity.

In special relativity it is enlightening to replace the parameter σ in (1) by one that is physically more meaningful. If the x-axis moves with constant velocity V relative to the x'-axis, for an observer on the x'-axis, $x'/t' = V$ is the constant velocity of the origin $x = 0$ of the x-axis.



But from (1) with $x = 0$

$$V = \frac{x'}{t'} = c\tanh\sigma,$$

so $\sinh\sigma = (V/c)/\sqrt{1-(V/c)^2}$ and $\cosh\sigma = 1/\sqrt{1-(V/c)^2}$. We can use this to rewrite the Lorentz transformation (1) in terms of the velocity V as

$$x' = \frac{x + Vt}{\sqrt{1-(V/c)^2}} \qquad t' = \frac{(V/c^2)x + t}{\sqrt{1-(V/c)^2}}.$$

It is physically obvious that to get the inverse transformation just replace V by $-V$.