Math 425, Spring 2011 **Jerry L. Kazdan**

Problem Set 9

DUE: Thursday April 7 [*Late papers will be accepted until 1:00 PM Friday*].

1. a) In a bounded region $\Omega \subset \mathbb{R}^n$, let $u(x,t)$ satisfy the modified heat equation

$$u_t = \Delta u + cu, \quad \text{where } c \text{ is a constant,} \tag{1}$$

as well as the initial and boundary conditions

$$u(x,0) = f(x), \quad \text{in } \Omega \quad \text{with } u(x,t) = 0 \text{ for } x \in \partial\Omega, \quad t \geq 0. \tag{2}$$

Let $u(x,t) = v(x,t)e^{\alpha t}$. Show that by picking the constant α cleverly, v satisfies equation (1) with $c = 0$ as well as (2).

Moral: one can easily reduce understanding equations (1)-(2) to the special case $c = 0$.

b) Generalize this to $u_t + a(t)u = \Delta u$ where $a(t)$ is any continuous function by seeking $u(x,t) = \varphi(t)v(x,t)$ and picking the function $\varphi(t)$ cleverly,

2. In a bounded region $\Omega \subset \mathbb{R}^n$, use the maximum principle to prove a uniqueness theorem for solutions $u(x,t)$ of the inhomogeneous equation

$$u_t - \Delta u = F(x,t) \quad \text{in } \Omega$$

with

$$u(x,0) = f(x), \quad \text{in } \Omega \quad \text{and } u(x,t) = \varphi(x,t) \text{ for } x \in \partial\Omega, \quad t \geq 0.$$

3. Let $\Omega \subset \mathbb{R}^n$ be a bounded region with smooth boundary $\partial\Omega$ and let $u(x,t)$ satisfy the heat equation

$$u_t = \Delta u \quad \text{for } x \in \Omega \quad \text{with initial temperature } u(x,0) = f(x).$$

If u satisfies Neumann boundary conditions $\partial u/\partial N = 0$ on $\partial\Omega$, show that

$$\lim_{t\to\infty} u(x,t) = \text{constant},$$

where the constant is the average of the initial temperature.

4. Let $u(x,t)$ be a solution of the heat equation $u_t = u_{xx}$ for $-1 < x < 1$, $t > 0$ with initial value $u(x,0) = 1 - x^2$ and boundary condition $u(\pm 1) = 0$.

a) Show that $0 < u(x,t) < 1$ for all $|x| < 1$ and $t > 0$.

b) Explain why $u(-x,t) = u(x,t)$ for all $-1 \leq x \leq 1$ and $t \geq 0$.

5. Let Ω be a bounded region in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and let $\varphi_k(x)$ and $\lambda_k > 0$, $k = 1, 2, 3, \ldots$ be the orthonormal eigenfunctions and corresponding eigenvalues for the Laplacian with zero Dirichlet boundary conditions:

$$-\Delta\varphi_k = \lambda_k \varphi_k \quad \text{in } \Omega, \qquad \varphi_k(x) = 0 \text{ for } x \in \partial\Omega.$$

Here we use the (real) inner product $\langle u, v\rangle := \iint_\Omega u(x)v(x)\,dx$.

a) Show that the solution of the inhomogeneous equation

$$-\Delta u = F(x) \quad \text{for } x \in \Omega, \qquad u(x) = 0 \text{ on } \partial\Omega,$$

is

$$u(x) = \sum_{k=1}^{\infty} \frac{\langle F, \varphi_k\rangle}{\lambda_k} \varphi_k(x).$$

b) Show this can be written as

$$u(x) = \iint_\Omega F(y)G(x,y)\,dy,$$

where

$$G(x,y) := \sum_{k=1}^{\infty} \frac{\varphi_k(x)\varphi_k(y)}{\lambda_k}$$

is called *Green's function* for this problem.

Bonus Problem

1-B Let $f(x)$ and $g(x)$ be 2π periodic functions with

$$0 < a \le f(x) \le b \qquad \text{and} \qquad 0 < \alpha \le g(x) \le \beta,$$

where a, b, α, β are constants. Assume $u(x)$ is a smooth 2π periodic solution of

$$-u''(x) = f(x) - g(x)e^{u(x)}.$$

Find constants m and M in terms of a, b, α, β so that

$$m \le u(x) \le M$$

for all x.

[Last revised: May 22, 2011]