Problem Set 4

DUE: In class Thursday, Feb. 17 *Late papers will be accepted until 1:00 PM Friday.*

1. Solve the wave equation $u_{tt} = c^2 u_{xx}$ for the semi-infinite string $x \geq 0$ with the initial and boundary conditions
$$u(x,0) = 3 - \sin x, \quad u_t(x,0) = 0, \quad u(0,t) = 3 - t^2.$$

2. [Weinberger p. 27 #3] Let $u(x,t)$ be a solution of the inhomogeneous wave equation $u_{tt} - c^2 u_{xx} = \sin \pi x$ for $0 < x < 1$, $t > 0$ with the boundary conditions $u(0,t) = 0$ and $u(1,t) = 0$.
   a) Find the solution if $u(x,0) = 0$ and $u_t(x,0) = 0$.
   b) Find the solution if $u(x,0) = x(1-x)$ and $u_t(x,0) = 0$.

3. Let $u(x,t)$ be the temperature at time t at the point x, $-1 \leq x \leq 1$. Assume it satisfies the heat equation $u_t = u_{xx}$ for $0 < t < \infty$ with the boundary condition $u(-1,t) = u(1,t) = 0$ and initial condition $u(x,0) = f(x)$.
   a) Show that $E(t) := \dfrac{1}{2}\displaystyle\int_{-1}^{1} u^2(x,t)\,dx$ is a decreasing function of t.
   b) Use this to prove uniqueness for the heat equation with these specified initial and boundary conditions.
   c) If $u(x,0) = f(x)$ is an even function of x, show that the temperature $u(x,t)$ at later times is also an even function of x

Bonus Problems

1-B [Generalization of #3] Let $u(x,y,t)$ is a solution of the heat equation $u_t = \Delta u$ in a bounded region $\Omega \subset \mathbb{R}^2$. Here one has the initial condition $u(x,y,0) = f(x,y)$ and boundary condition $u(x,y,t) = 0$ at points (x,y) on the boundary, $\partial\Omega$ and define
$$E(t) := \frac{1}{2}\iint_{\Omega} u^2(x,y,t)\,dx\,dy.$$

   a) Generalize #3(a)(b) to this setting.
   b) If Ω is symmetric under refection across the y-axis, so $(x,y) \to (-x,y)$ and if the initial temperature is also symmetric, $f(x,y) = f(-x,y)$, show that $u(x,y,t) = u(-x,y,t)$ for all points in Ω and all $t > 0$.
   REMARK: If Ω is the unit disk, $\{x^2 + y^2 < 1\}$ and $f(x,y)$ depends only the distance to the origin, $r := \sqrt{x^2 + y^2}$, the same reasoning shows that the solution u also depends only on r.

[Last revised: March 9, 2011]