Math 425/525, Spring 2011 **Jerry L. Kazdan**

Problem Set 3

DUE: In class Thursday, Feb. 10 *Late papers will be accepted until 1:00 PM Friday.*

1. Solve $u_x + u_y + u = e^{x+2y}$ with $u(x,0) = 0$.

2. Find the general solution of $u_{xy} = x^2 y$ for the function $u(x,y)$.

3. Find the general solution of the inhomogeneous equation $u_{tt} - u_{xx} = 1 + 2x$ for the function $u(x,t)$, where $-\infty < x < \infty$ (an infinite string).

4. Solve the wave equation (for an infinite string) $u_{tt} = c^2 u_{xx}$ with initial conditions $u(x,0) = \ln(1+x^2)$ and $u_t(x,0) = 4 + x$.

5. [Weinberger, p.17 #3] A string of length $L = 1$ with fixed end points is initially fixed in the position $u(x,0) = \sin \pi x$ and is released at time $t = 0$ (so its initial velocity is zero). Find its subsequent motion.

6. [THE DULCIMER] Solve the wave equation $u_{tt} = c^2 u_{xx}$ with initial conditions $u(x,0) = 0$ and $u_t(x,0) = g(x)$, where $g(x) = 1$ if $|x| < a$ and $g(x) = 0$ for $|x| \ge a$. This corresponds to hitting the string with a hammer of width $2a$. Draw sketches of snapshots of the string (i.e., plot u versus x) for $t = \frac{1}{2}a/c$, $t = a/c$, $t = \frac{3}{2}a/c$, $t = 2a/c$ and $t = \frac{5}{2}a/c$.

7. On the bounded interval $0 \le x \le L$, let $u(x,t)$ be a solution of the wave equation for a vibrating string: $u_{tt} = u_{xx}$. Define the "Energy" by

$$E(t) := \frac{1}{2}\int_0^L [u_t^2 + u_x^2]\, dx. \tag{1}$$

   Assume u satisfies the boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$.

   a) Show that $E(t)$ is a constant, so energy is conserved.

   b) [Uniqueness] Say the two functions $v(x,t)$ and $w(x,t)$ both satisfy the wave equation and boundary conditions and *also* have the same initial position and velocity:

$$v(x,0) = w(x,0) \qquad v_t(x,0) = w_t(x,0).$$

   Show that $v(x,t) = w(x,t)$ for all $0 \le x \le L$, $t \ge 0$.

8. Take a moment to review the divergence theorem from vector calculus, then work the following problem:

   Suppose $V(x,y,z)$ is a vector-valued function defined everywhere in 3-dimensional space. Further, suppose that V is differentiable and that

$$\|V(x,y,z)\| \leq \frac{1}{1+(x^2+y^2+z^2)^{3/2}}$$

   for all (x,y,z). Show that

$$\iiint_{\mathbb{R}^3} \nabla\cdot V(x,y,z)\,dx\,dy\,dz = 0. \tag{2}$$

   REMARK Let $B(0,R)$ be the ball of radius R centered at the origin. Then (2) means that

$$\lim_{R\to\infty} \iiint_{B(0,R)} \nabla\cdot V(x,y,z)\,dx\,dy\,dz = 0.$$

Bonus Problems (Due Feb. 10)

1-B WAVE EQUATION WITH FRICTION If there is friction, the wave equation becomes

$$u_{tt} + bu_t = u_{xx}, \qquad \text{where} \qquad b \geq 0.$$

Consider a string of length L whose ends are fixed: $u(0,t) = 0$ and $u(L,t) = 0$ for all $t \geq 0$. Define the energy using (1).

a) Show that $dE/dt \leq 0$.

b) Use this to show that the uniqueness assertion in problem 7b) of the previous problem is still true.

2-B Solve $u_{xx} + u_{xt} - 20u_{tt} = 0$ with initial conditions $u(x,0) = \varphi(x)$, and $u_t(x,0) = \psi(x)$.

[Last revised: February 11, 2011]