Math 425 **Dr. DeTurck**
Assignment 4 **Now due Tuesday, February 23, 2010**

Reading: Textbook, Chapter 3.

1. Consider two homogeneous rods, one of length L_1 and the other of length L_2. Suppose that both rods have the same density ρ, the same cross section and the same specific heat constant c. But they have different heat conductivities, κ_1 and κ_2 respectively. Their heat diffusion constants are $k_1 = \kappa_1/(c\rho)$ and $k_2 = \kappa_2/(c\rho)$. They are welded together so that the temperature u and the heat flux κu_x across the weld are continuous. The left end of the left-hand rod (the one with diffusion constant k_1) is maintained at temperature zero, and the right end of the right-hand rod is maintained at T degrees.

   (a) Find the equilibrium temperature distribution in the composite rod. (b) Sketch the equilibrium temperature distribution as a function of x, assuming $L_1 = 3$, $L_2 = 2$, $k_1 = 2$, $k_2 = 1$ and $T = 10$.

2. Consider the Neumann problem for the Laplace equation on a domain $D \subset R^3$:

$$\Delta u = f(x, y, z) \text{ in } D$$

$$\frac{\partial u}{\partial n} = 0 \text{ on the boundary of } D.$$

   (a) What solutions of this problem are there other than $u = 0$ if $f = 0$? Why does this mean that the solution of the above problem is not unique even if f is not zero?

   (b) Explain why

$$\iiint_D f(x, y, z)\, dx\, dy\, dz = 0$$

   is a necessary condition for the Neumann problem to have a solution (divergence theorem!).

   (c) Give a physical interpretation of (a) and (b) for heat flow or diffusion.

3. Consider the heat equation $u_t = u_{xx}$ in $\{0 < x < 1,\ 0 < t < \infty\}$ with $u(0, t) = u(1, t) = 0$ and $u(x, 0) = 4x(1 - x)$.

   (a) Explain why $0 < u(x, t) < 1$ for all $t > 0$ and $0 < x < 1$.

   (b) Explain why $u(x, t) = u(1 - x, t)$ for all $t \geq 0$ and $0 \leq x \leq 1$.

   (c) Show that $\int_0^1 u^2\, dx$ is a strictly decreasing function of t.

4. Prove the comparison principle for the heat equation: If u and v are solutions of the heat equation and $u(x, 0) \leq v(x, t)$ for all $x \in [0, L]$ and $u(0, t) \leq v(0, t)$ and $u(L, t) \leq v(L, t)$ for all $t \geq 0$, then $u(x, t) \leq v(x, t)$ for all $(x, t) \in [0, L] \times [0, \infty)$.

5. Solve the heat equation $u_t = k u_{xx}$ with $u(x, 0) = e^{3x}$.

6. Solve the following variant of the heat equation

$$u_t - k u_{xx} + bt^2 u = 0$$

   for $-\infty < x < \infty$ with $u(x, 0) = f(x)$, where $b > 0$ is a constant. (Hint: The solutions of the related ordinary differential equation $w' + bt^2 w = 0$ are $ce^{-bt^3/3}$. So try making the change of variables $u(x, t) = e^{-bt^3/3} v(x, t)$ and derive [and solve] an equation for v.)