PDE: Linear Change of Variable

Let $x := (x_1, x_2, \ldots, x_n)$ be a point in $\mathbb{R}^n$ and consider the second order linear partial differential operator

$$Lu := \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}, \tag{1}$$

where the coefficient matrix $A := (a_{ij})$ is constant. Since for functions whose second derivatives are continuous we know that

$$\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_i}\right)$$

we may (and will) assume that A is a symmetric matrix: $A = A^*$.

In these brief notes we obtain a useful formula for how L changes if we make the linear change of variable $y = Sx$ where $(S := s_{k\ell})$ is a constant matrix. Written in coordinates this means that

$$y_k = \sum_{\ell=1}^{n} s_{k\ell} x_\ell, \quad \text{where} \quad k = 1, \ldots, n.$$

FIRST GOAL: Compute L in these new y coordinates. This is straightforward (even boring) if you just keep calm and don't make copying errors. By the chain rule

$$\frac{\partial u}{\partial x_j} = \sum_{k=1}^{n} \frac{\partial u}{\partial y_k}\frac{\partial y_k}{\partial x_i} = \sum_{k=1}^{n} \frac{\partial u}{\partial y_k} s_{ki}. \tag{2}$$

We repeat this process to compute the second derivatives:

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_j}\right) = \sum_{\ell=1}^{n} \frac{\partial}{\partial y_\ell}\left(\qquad \right)\frac{\partial y_\ell}{\partial x_j} = \sum_{\ell=1}^{n} \frac{\partial}{\partial y_\ell}\left(\qquad \right) s_{\ell j},$$

so using (2)

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_{\ell=1}^{n} \frac{\partial}{\partial y_\ell}\left(\sum_{k=1}^{n} \frac{\partial u}{\partial y_k} s_{ki}\right) s_{\ell j} = \sum_{k,\ell=1}^{n} \frac{\partial^2 u}{\partial y_k \partial y_\ell} s_{ki} s_{\ell j}.$$

Consequently

$$Lu = \sum_{i,j=1}^{n} a_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_{k,\ell=1}^{n} \left[\sum_{i,j=1}^{n} a_{i,j} s_{ki} s_{\ell j}\right] \frac{\partial^2 u}{\partial y_k \partial y_\ell}$$

so

$$Lu = \sum_{k,\ell=1}^{n} b_{k\ell} \frac{\partial^2 u}{\partial y_k \partial y_\ell}, \tag{3}$$

where the coefficient matrix $B := (b_{k\ell})$ is

$$b_{k\ell} = \sum_{i,j=1}^{n} a_{i,j} s_{ki} s_{\ell j}.$$

In terms of matrices this simply says that

$$B = SAS^*. \tag{4}$$

SECOND GOAL: Pick the matrix S defining the change of coordinates $y = Sx$ to make (3) as simple as possible. We'll be able to make B into a diagonal matrix by diagonalizing A Since A is a symmetric matrix, there is an orthogonal matrix R that diagonalizes it (in $\mathbb{R}^n$, an orthogonal matrix is just the generalization of a rotation). Thus

$$R^{-1}AR = \Lambda = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

Thus $S = R\Lambda R^{-1}$. Since for an orthogonal matrix R we know that $R^* = R^{-1}$, is we let $S = R^*$, then $\Lambda = SAS^*$. Comparing with (4) we see that using this change of coordinates we have arranged that B is a diagonal matrix.

Consequently L has the much simpler form

$$Lu = \lambda_1 \frac{\partial^2 u}{\partial y_1^2} + \lambda_2 \frac{\partial^2 u}{\partial y_2^2} + \cdots + \lambda_n \frac{\partial^2 u}{\partial y_n^2}. \tag{5}$$

We can make one further simplification. By stretching the coordinates to have the coefficients in (5) be either 1, 0, or -1. For instance, if $\lambda_1 > 0$, replace y_1 by the new stretched coordinate $z_1 := y_1/\sqrt{\lambda_1}$. As an example, using this device

$$Lu := 4\frac{\partial^2 u}{\partial y_1^2} - 9\frac{\partial^2 u}{\partial y_2^2} \quad \text{becomes} \quad Lu := \frac{\partial^2 u}{\partial z_1^2} - \frac{\partial^2 u}{\partial z_2^2}.$$

EXERCISE: Show that there is a linear change of variable so that at one point, say the origin, the second derrivative matrix

$$\frac{\partial^2 u}{\partial y_k \partial y_\ell}(0)$$

is a diagonal matrix.