Math 425, Spring 2011 **Jerry L. Kazdan**

Inner Product Summary

This is a summary of some items from class on Tues, Feb. 15, 2011.

SETTING: Linear spaces X, Y with inner products $\langle\ ,\ \rangle_X$ and $\langle\ ,\ \rangle_Y$.

Example: $X = \mathbb{R}^4$ and $Y = \mathbb{R}^7$.

Vectors $x,\ z \in X$ are *orthogonal* if $\langle x, z\rangle_X = 0$.

Let $L : X \to Y$ be a linear map. Then the *adjoint map* $L^* : Y \to X$ is defined by the property

$$\langle Lx, y\rangle_Y = \langle x, L^*y\rangle_X \quad \text{for all} \quad x \in X,\ y \in Y.$$

Observation: $(LM)^* = M^*L^*$.

For real matrices, the adjoint is just the transpose. For complex matrices, it is complex conjugate transpose.

Instead of writing $\langle\ ,\ \rangle_X$ etc, we'll write $\langle\ ,\ \rangle$ since the inner product being used will be obvious.

In $L_2(a,b)$ on functions f with $f(a) = 0$ and $f(b) = 0$, if $L := \frac{d}{dx}$, then $L^* = -\frac{d}{dx}$. If one ignores the boundary conditions (that is, forget the boundary terms when integrating by parts), one gets the *formal adjoint*.

PROJECTION AND ORTHOGONAL DECOMPOSITION. Let $V \subset X$ be a linear subspace. If $x \in X$, write

$$x = v + z, \quad \text{where} \quad v \in V, \quad z \perp V.$$

We write $v = P_V x$ and call it the *orthogonal projection of x into* V. $P : X \to X$ is a linear map that satisfies $P^2 = P$ and $P = P^*$. Note that $z = x - v = (I - P_v)x$. Also $\|x\|^2 = \|v\|^2 + \|z\|^2$.
Let $e_1, e_2, \dots e_N$ be an orthonormal basis for V (this assumes V is finite dimensional). then any $x \in V$ can be written (uniquely) as

$$x = a_1 e_1 + \cdots + a_N e_N, \quad \text{where} \quad a_k = \langle x, e_k \rangle,$$

Consequently for any $x \in X$, we have

$$P_v x = a_1 e_1 + \cdots + a_N e_N, \quad \text{where} \quad a_k = \langle x, e_k \rangle,$$

and the Pythagorean formula

$$\|P_v x\|^2 = |a_1|^2 + |a_2|^2 + \cdots + |a_N|^2.$$

EXAMPLE: $X = \mathbb{R}^3$, and we write $x = (x_1, x_2, x_3)$, an example with V points of the form $V = (x_1, 0, x_3)$ is $P_V x = (x_1, 0, x_3)$.

EXAMPLE: FOURIER SERIES Here $X = L_2(-\pi, \pi)$,

$$V_N = \operatorname{span}\{1, \cos x, \cos 2x, \dots, \cos Nx,\ \sin x, \dots, \sin Nx\}.$$

An orthonormal basis is:

$$e_0 := \frac{1}{\sqrt{2\pi}},\ e_1 := \frac{\cos x}{\sqrt{\pi}}, \dots, e_N := \frac{\cos Nx}{\sqrt{\pi}}, \varepsilon_1 := \frac{\sin x}{\sqrt{\pi}}, \dots, \varepsilon_N := \frac{\sin Nx}{\sqrt{\pi}}.$$

We want to write the projection of $f(x)$ into V_N, so

$$\begin{aligned} P_{V_N} f(x) &= a_0 e_0 + (a_1 e_1 + \cdots + a_N e_N) + (b_1 \varepsilon_1 + \cdots + b_N \varepsilon_N) \\ &= a_0 \frac{1}{\sqrt{2\pi}} + \left(a_1 \frac{\cos x}{\sqrt{\pi}} + \cdots + a_N \frac{\cos Nx}{\sqrt{\pi}}\right) + \left(b_1 \frac{\sin x}{\sqrt{\pi}} + \cdots + b_N \frac{\sin N}{\sqrt{\pi}}\right. \\ &= a_0 \frac{1}{\sqrt{2\pi}} + \sum_{k=1}^{N} \left[a_k \frac{\cos kx}{\sqrt{\pi}} + b_k \frac{\sin kx}{\sqrt{\pi}}\right]. \end{aligned}$$

so

$$f(x) = P_{V_N} f(x) + h_N(x) = a_0 \frac{1}{\sqrt{2\pi}} + \sum_{k=1}^{N} \left[a_k \frac{\cos kx}{\sqrt{\pi}} + b_k \frac{\sin kx}{\sqrt{\pi}} \right] + h_N(x)$$

where $h_N := f - P_V f$ is automatically orthogonal to V_N.

The Pythagorean formula gives

$$\|f\|^2_{L_2(-\pi.\pi)} = |a_0|^2 + \sum_{k=1}^{N} \left(|a_k|^2 + |b_k|^2 \right) + \|h_N\|^2_{L_2(-\pi.\pi)}. \qquad (1)$$

Of course, one hopes that $\lim_{N\to\infty} \|h_N\|_{L_2(-\pi.\pi)} = 0$. It is true for essentially all functions – certainly for all piecewise continuous functions f.

[Last revised: February 17, 2011]