Signature   PRINTED NAME

My signature above certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this examination.

Math 425   **Exam 2**   Jerry L. Kazdan
April 26, 2011   12:00 – 1:20

DIRECTIONS This exam has three parts, Part A, short answer, has 1 problem (10 points). Part B has 4 shorter problems (9 points each, so 36 points). Part C has 3 traditional problems (15 points each so 45 points). Total is 91 points.
Closed book, no calculators or computers– but you may use one $3'' \times 5''$ card with notes on both sides.

Part A: Short Answer (1 problem, 10 points).

1. Let S and T be linear spaces and $A : S \to T$ be a linear map. Say $\mathbf{V}$ and $\mathbf{W}$ are particular solutions of the equations $A\mathbf{V} = \mathbf{Y}_1$ and $A\mathbf{W} = \mathbf{Y}_2$, respectively, while $\mathbf{Z} \neq 0$ is a solution of the homogeneous equation $A\mathbf{Z} = 0$.

   Answer the following in terms of $\mathbf{V}$, $\mathbf{W}$, and $\mathbf{Z}$.

   a) Find some solution of $A\mathbf{X} = 3\mathbf{Y}_1$.
   b) Find some solution of $A\mathbf{X} = -5\mathbf{Y}_2$.
   c) Find some solution of $A\mathbf{X} = 3\mathbf{Y}_1 - 5\mathbf{Y}_2$.
   d) Find another solution (other than $\mathbf{Z}$ and 0) of the homogeneous equation $A\mathbf{X} = 0$.
   e) Find another solution of $A\mathbf{X} = 3\mathbf{Y}_1 - 5\mathbf{Y}_2$.

Part B: Short Problems (4 problems, 9 points each so 36 points)

B–1. Suppose f is a function of one variable that has a continuous second derivative. Show that for any constants a and b, the function

$$u(x, y) = f(ax + by)$$

is a solution of the nonlinear PDE

$$u_{xx}u_{yy} - u_{xy}^2 = 0.$$

B–2. $\mathbf{U} = (1, 1, 0, 1)$ and $\mathbf{V} = (-1, 2, 0, -1)$ are orthogonal vectors in R^4.

Write the vector $\mathbf{X} = (1, 1, 1, 0)$ in the form $\mathbf{X} = a\mathbf{U} + b\mathbf{V} + \mathbf{W}$, where a, b are scalars and $\mathbf{W}$ is a vector perpendicular to $\mathbf{U}$ and $\mathbf{V}$.

B–3. If $u(x, y)$ is a solution of the Laplace equation in the unit disk $x^2 + y^2 < 1$ with boundary conditions

$$u(x, y) = \begin{cases} 1 & \text{for } x^2 + y^2 = 1, \quad y > 0 \\ 0 & \text{for } x^2 + y^2 = 1, \quad y \leq 0. \end{cases}$$

Compute $u(0, 0)$.

B–4. This problem concerns the solution of the initial-value problem for the wave equation $u_{tt} = u_{xx} + u_{yy}$ in two space variables $(x, y) \in \mathbb{R}^2$, together with the initial conditions

$$u(x, y, 0) = f(x, y), \qquad u_t(x, y, 0) = 0.$$

If $f(x, y)$ is a 2π periodic functions of x, so $f(x+2\pi, y) = f(x, y)$ for all x, show that $u(x, y, t)$ is also a 2π periodic function of x.

Part C: Traditional Problems (3 problems, 15 points each so 45 points)

C–1. Let $\Omega \subset \mathbb{R}^2$ be a bounded region in the plane.

a) Let $w(x, y, t)$ be a solution of the modified heat equation

$$w_t = w_{xx} + w_{yy} - 7w_x + w_y - 5w$$

for $(x, y) \in \Omega$ and $0 < t \le T < \infty$. Show that the solution w cannot have a local positive maximum or negative minimum at a point of Ω.

NOTE: There are two cases, one where the maximum point accurs at a point (x, y, t) with $0 < t < T$ and one at a point (x, y, T)

b) If $w(x, y, 0) = \sin(x + 2y)$ for $(x, y) \in \Omega$ and $-2 \le w(x, y, t) \le 3$ for $(x, y) \in \partial\Omega$, $t \ge 0$, what can you conclude about the size of $w(x, y, t)$ for $(x, y) \in \Omega$, $t \ge 0$?.

C–2. In a bounded region $\Omega \subset \mathbb{R}^n$, let $u(x, t)$ satisfy the modified heat equation

$$u_t - 2tu = \Delta u, \tag{1}$$

as well as the initial and boundary conditions

$$u(x, 0) = f(x), \quad \text{in } \Omega \quad \text{with } u(x, t) = 0 \text{ for } x \in \partial\Omega, \quad t \ge 0. \tag{2}$$

Let $u(x, t) = \varphi(t)v(x, t)$. Show that by picking the function $\varphi(t)$ cleverly, v satisfies the standard heat equation $v_t = \Delta v$ as well as the initial and boundary conditions (2).

REMARK: This generalized to $u_t + a(t)u = \Delta u$ where $a(t)$ is any continuous function.

C–3. The motion $u(x, y, t)$ of a special drum $\Omega \in \mathbb{R}^2$ satisfile the modified wave equation

$$u_{tt} + b(x, y, t)u_t = \Delta u \quad \text{for } (x, y) \in \Omega, \quad t > 0. \tag{3}$$

with boundary condition

$$u(x, y, t) = 0 \quad \text{for } (x, y) \in \partial\Omega, \ t \ge 0. \tag{4}$$

Define the "energy"

$$E(t) := \frac{1}{2} \iint_\Omega \left[u_t^2 + |\nabla u|^2\right] dx\, dy.$$

Assume that $|b(x, y, t)| \le m$ for some constant m and all $(x, y) \in \Omega$, $t \ge 0$.

a) Show that $\dfrac{dE}{dt} \le 2mE$ for all $t \ge 0$.

b) Deduce that $\dfrac{d}{dt}\left[e^{-2mt}E(t)\right] \le 0$ for all $t \ge 0$, and hence that

$$E(t) \le e^{2mt}E(0) \quad \text{for all} \quad t \ge 0.$$

c) If $u(x, y, 0) = 0$ and $u_t(x, y, 0) = 0$ for $(x, y) \in \Omega$, what does this say about $E(t)$ for $t \ge 0$ and hence about $u(x, y, t)$ for $t \ge 0$?