Math 425 **Exam 1** Jerry L. Kazdan
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DIRECTIONS This exam has three parts, Part A, short answer, has 1 problem (12 points). Part B has 5 shorter problems (7 points each, so 35 points). Part C has 3 traditional problems (15 points each so 45 points). Total is 92 points.
Closed book, no calculators or computers– but you may use one $3'' \times 5''$ card with notes on both sides.

Part A: Short Answer (1 problems, 12 points).

1. Let S and T be linear spaces and $A : S \to T$ be a linear map. Say $\mathbf{V}$ and $\mathbf{W}$ are particular solutions of the equations $A\mathbf{V} = \mathbf{Y}_1$ and $A\mathbf{W} = \mathbf{Y}_2$, respectively, while $\mathbf{Z} \neq 0$ is a solution of the homogeneous equation $A\mathbf{Z} = 0$.

   Answer the following in terms of $\mathbf{V}$, $\mathbf{W}$, and $\mathbf{Z}$.

   a) Find some solution of $A\mathbf{X} = 3\mathbf{Y}_1$. SOLUTION: $\mathbf{X} = 3\mathbf{V}$

   b) Find some solution of $A\mathbf{X} = -5\mathbf{Y}_2$. SOLUTION: $\mathbf{X} = -5\mathbf{W}$

   c) Find some solution of $A\mathbf{X} = 3\mathbf{Y}_1 - 5\mathbf{Y}_2$. SOLUTION: $\mathbf{X} = 3\mathbf{V} - 5\mathbf{W}$

   d) Find another solution (other than $\mathbf{Z}$ and 0) of the homogeneous equation $A\mathbf{X} = 0$. SOLUTION: $\mathbf{X} = 2\mathbf{Z}$

   e) Find *two* solutions of $A\mathbf{X} = \mathbf{Y}_1$. SOLUTION: $\mathbf{X} = \mathbf{V}$ and $\mathbf{X} = \mathbf{V} + \mathbf{Z}$

   f) Find another solution of $A\mathbf{X} = 3\mathbf{Y}_1 - 5\mathbf{Y}_2$. SOLUTION: $\mathbf{X} = 3\mathbf{V} - 5\mathbf{Y}_2 + \mathbf{Z}$

Part B: Short Problems (5 problems, 7 points each so 35 points)

B–1. $\mathbf{U} = (1, 1, 0, 1)$ and $\mathbf{V} = (-1, 2, 1, -1)$ are orthogonal vectors in R^4.

Write the vector $\mathbf{X} = (1, 1, 1, 2)$ in the form $\mathbf{X} = a\mathbf{U} + b\mathbf{V} + \mathbf{W}$, where a, b are scalars and $\mathbf{W}$ is a vector perpendicular to $\mathbf{U}$ and $\mathbf{V}$.

SOLUTION: Since $\|\mathbf{U}\| = \sqrt{\mathbf{3}}$ and $\|\mathbf{V}\| = \sqrt{\mathbf{7}}$, then $\hat{\mathbf{U}} := \mathbf{U}/\sqrt{3}$ and $\hat{\mathbf{V}} := \mathbf{V}/\sqrt{7}$ are orthonormal vectors in the same directions as $\mathbf{U}$ and $\mathbf{V}$, respectively. We'll write $\mathbf{X}$ in the form

$$\mathbf{X} = \alpha\hat{\mathbf{U}} + \beta\hat{\mathbf{V}} + \mathbf{W}, \qquad (1)$$

where $\mathbf{W}$ is a vector perpendicular to both $\hat{\mathbf{U}}$ and $\hat{\mathbf{V}}$ and hence $\mathbf{U}$ and $\mathbf{V}$.

Taking the inner product of both sides if (1) with $\hat{\mathbf{U}}$ and $\hat{\mathbf{V}}$ we find that

$$\alpha = \langle \mathbf{X}, \hat{\mathbf{U}} \rangle = \frac{\mathbf{4}}{\sqrt{\mathbf{3}}} \quad \text{and } \beta = \langle \mathbf{X}, \hat{\mathbf{V}} \rangle = \mathbf{0}.$$

Thus

$$\mathbf{X} = \frac{4}{\sqrt{3}}\hat{\mathbf{U}} + \mathbf{W} = \frac{4}{3}\mathbf{U} + \mathbf{W},$$

where $\mathbf{W}$ is defined by this equation. It is orthogonal to both $\mathbf{U}$ and $\mathbf{V}$ since that is how we computed α and β.

B–2. Find $u(x,t)$ that satisfies $u_x - 2u_t = 1$ with $u(x,0) = 0$.

SOLUTION: By naively guessing, a particular solution of the infomogeneous equation is $u_{\text{part}} := x$. The general solution of the homogeneous equation is $u(x,t)_{\text{hom}} := f(2x+t)$ for any (differentiable) function f. Thus the general solution of $u_x - 2u_t = 1$ is

$$u(x,t) = f(2x+t) + x.$$

Now match the initial conditions: $0 = u(x,0) = f(2x) + x$ so $f(x) = -x/2$. Thus the desired solution is

$$u(x,t) = -(2x+t)/2 + x = -t/2.$$

To guard against errors, it is important to verify that this works (it does).

B–3. Let $u(x,t)$ be a solution of the wave equation

$$u_{tt} = 4u_{xx}, \quad \text{for} \quad -\infty < x < \infty,\ t \geq 0,$$

with the (continuous) initial conditions

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x).$$

Find the largest interval $J = \{a \leq x \leq b\}$ where changing $f(x)$ or $g(x)$ at any point of J can change ("influence") the value of $u(0,3)$. In other words, in the (x,t) plane, find all the points on the x-axis that are in the domain of dependence of $(0,\ 3)$.

SOLUTION: For the general equation $u_{tt} = c^2 u_{xx}$, to find the domain of dependence of a point $P := (x_0, t_0)$, draw the lines $x - ct = \text{const}_1$ and $x + ct = \text{const}_2$ that go through P. The domain of dependence are the points (x,t) the region ("backward cone") between these lines with $t \leq t_0$.

In this particular problem, these lines are $x - 2t = x_0 - 2t_0 = -6$ and $x + 2t = x_0 + 2t_0 = 6$. The initial conditions are placed on the line where $t = 0$. Thus the points in the domain of dependence at $t = 0$ is the interval $-6 \leq x \leq 6$.

B–4. Find the general solution $u(x,y)$ of $u_{xy} = 4y$.

SOLUTION: By integrating twice, it is obvious that a particular solution of the inhomogeneous equation is $u_{\text{part}} = 2xy^2$. The general solution, $v(x,y)$, of the homogeneous equation is

$$v(x,y) = \varphi(x) + \psi(y)$$

for any (differentiable) functions $\varphi(x)$ and $\psi(y)$. Thus the general solution of the inhomogeneous equation is

$$u(x,y) = \varphi(x) + \psi(y) + 2xy^2.$$

B–5. Let $u(x,y)$ and $v(x,y)$ be a solutions of the Laplace equation $\Delta u = 0$, $\Delta v = 0$ in a bounded region Ω in the plane. If $u > v$ on the boundary of Ω, what, if anything, can you conclude about the relationship between u and v inside Ω? Justify your assertion.

SOLUTION: Let $w := u - v$. Then $\Delta w = 0$ in Ω and $w > 0$ on the boundary of Ω. Thus, by the maximum principle $w > 0$ throughout Ω, that is, $u > v$ throughout Ω.

Part C: Traditional Problems (3 problems, 15 points each so 45 points)

C–1. Find the motion $u(x,t)$ of a clamped string $\{0 \le x \le \pi\}$

$$u_{tt} = u_{xx},$$

with initial and boundary conditions:

$$u(x,0) = 0, \quad u_t(x,0) = 15\sin 5x, \quad \text{and} \quad u(0,t) = u(\pi,t) = 0.$$

SOLUTION: As usual, use separation of variables and seek special solutions of the form $u(x,t) = X(x)T(t)$. Substituting in the wave equation this gives

$$\frac{X''(x)}{X(x)} = \frac{\ddot{T}(t)}{T(t)} = \text{const} = \alpha,$$

so

$$X'' - \alpha X = 0 \quad \text{and } \ddot{T} - \alpha T = 0.$$

To match the boundary conditions $u(0,t) = u(\pi,t) = 0$ we need $\alpha = -k^2$, $k = 1, 2, \dots$ and find the special solutions

$$u_k(x,t) = [A_k \cos kt + B_k \sin kt] \sin kx, \quad k = 1, 2, \dots,$$

so

$$u(x,t) = \sum_{k=1}^{\infty} [A_k \cos kt + B_k \sin kt] \sin kx,$$

where the coefficients A_k and B_k are found by matching the initial conditions:

$$u(x,0) = \sum_{k=1}^{\infty} A_k \sin kx, \qquad u_t(x,0) = \sum_{k=1}^{\infty} kB_k \sin kx.$$

For this problem the initial data is so simple that by inspection $A_k = 0$ for all k and $B_k = 0$ for all k *except* $5B_5 = 15$, so $B_5 = 3$. Thus

$$u(x,t) = 3\sin 5t \, \sin 5x.$$

As a guard against errors, it is easy to verify that this satisfies all the required conditions.

C–2. Let $u(x,y)$ satisfy $\Delta u - u = 0$ in a bounded region $\Omega \subset \mathbb{R}^2$ with $u(x,y) = 0$ on the boundary of Ω. Use Green's identity to show that $u(x,y) = 0$ throughout Ω.

SOLUTION: Green's First identity states:

$$\iint_\Omega \varphi \Delta\psi \, dx\, dy = \int_{\partial\Omega} \varphi \frac{\partial \psi}{\partial N}\, ds - \iint_\Omega \nabla\varphi \cdot \nabla\psi \, dx\, dy$$

for all twice continuously differentiable functions $\psi(x,y)$, $\varphi(x,y)$. Applying this with $\varphi = \psi = u$ yields

$$\iint_\Omega u^2 \, dx\, dy = -\iint_\Omega |\nabla u|^2 \, dx\, dy \le 0,$$

so $u(x,y) \equiv 0$ throughout Ω.

This same reasoning applies to solutions of $\Delta u - c(x,y)u = 0$, if we assume that $c(x,y) \ge 0$.

C–3. Let $u(x,t)$ be the temperature of a rod of length L that satisfies

$$u_t = u_{xx} - ru \qquad \text{for } 0 < x < L, \quad t > 0,$$

where $r > 0$ is a constant [this is related to the heat equation but assumes that heat radiates out into the air along the rod]. Assume u satisfies the initial condition $u(x,0) = f(x)$.

Define the total heat energy by $E(t) = \frac{1}{2}\int_0^L u^2(x,t)\, dx$.

a) If u also satisfies the Dirichlet boundary conditions

$$u(0,t) = 0, \qquad u(L,t) = 0$$

(the ends of the rod are held at temperature 0), show that $E(t)$ is a decreasing function of t.

SOLUTION: Use the PDE and integrate by parts:

$$\frac{dE}{dt} = \int_0^L u u_t \, dx = \int_0^L u[u_{xx} - ru]\, dx = u u_x\Big|_{x=0}^{L} - \int_0^L [u_x^2 + ru^2]\, dx \le 0 \qquad (2)$$

b) Show that even if u satisfies Neumann boundary conditions

$$u_x(0,t) = 0, \qquad u_x(L,t) = 0$$

(the ends of the rod are insulated), $E(t)$ is still a decreasing function of t.

SOLUTION: The previous computation (2) still works.

c) [Extra credit!] Show that in either of the above cases $\lim_{t\to\infty} E(t) = 0$.

SOLUTION: Notice that (2) has the stronger consequence

$$\frac{dE}{dt} = -\int_0^L [u_x^2 + ru^2]\, dx \le -r \int_0^L u^2\, dx = -2rE,$$

that is, $E' + 2rE \le 0$, so $[e^{2rt}E(t)]' \le 0$. In words, $e^{2rt}E(t)$ is a decreasing (really, only "non-increasing") function. Consequently,

$$e^{2rt}E(t) \le E(0) \quad \text{for all} \quad t \ge 0.$$

Thus

$$E(t) \le e^{-2rt}E(0) \to 0 \quad \text{as} \quad t \to \infty.$$